

EQUIVOCATION IN THE FOUNDATIONS OF LEIBNIZ'S INFINITESIMAL FICTIONS

Tzuchien THO*

Abstract. In this article, I address two different kinds of equivocations in reading Leibniz's fictional infinite and infinitesimal. These equivocations form the background of a reductive reading of infinite and infinitesimal fictions either as ultimately finite or as something whose status can be taken together with any other mathematical object as such. The first equivocation is the association of a foundation of infinitesimals with their ontological status. I analyze this equivocation by criticizing the logicist influence on 20th century Anglophone reception of the syncategorematical infinite and infinitesimal. The second equivocation is the association of the rigor of mathematical demonstration with the problem of the admissibility of infinite or infinitesimal terms. I analyze this by looking at Leibniz's constructive method and apagogic argument style in his quadrature method. In treating these equivocations, I critique some assumptions that underlie the reductive reading of Leibniz's fictionalism concerning infinite and infinitesimals. In turn, I suggest that these infinitesimal "fictions" pointed to a problematic within Leibniz's work that was conceived and reconsidered in Leibniz's work from a range of different contexts and methods.

Keywords: Leibniz, differential calculus, infinitesimal, foundations of mathematics, quadrature, apagogic proofs in mathematics

Introduction

Through the painstaking work of many archivists, historians and philosophers in the last few decades, a great deal of what was unclear about Leibniz's infinitesimals has been cleared up. Series VII of the Akademie edition has since 1990 made available works that demonstrate not only the degree of Leibniz's involvement in a wide range of mathematical problems but also a deeper appreciation of the complexity of his methodology in the development of the differential calculus. This capacity to look more closely at Leibniz's mathematical methodology is important precisely in understanding the concrete aspects of these "fictional" infinitesimals beyond the mere determination of these terms as syncategorematical; a certainty that one could already

* Berlin-Brandenburgische Akademie der Wissenschaften, Emserstrasse 93, 12051 Berlin, email: tzuchien.tho@gmail.com

grasp from his *New Essays on Human Understanding* and other previously available correspondences.

What remains unclear, despite all that has been revealed, is the meaning of this syncategorematical infinite, the role that it plays within Leibniz's mathematical reflections and its importance for his larger metaphysical and scientific considerations. As many interpreters have noted, a syncategorematical use of the infinitesimal is, as Leibniz himself seems to suggest, the use of an expression that can be reduced to more standard terms. As Leibniz says to Pinsson in a letter of 29 August 1701, "[I]n lieu of the infinite or infinitely small, we take quantities as great or as small as it is required so that the error would be less than the given error such that we do not differ from the style of Archimedes except in the expressions [...]."¹ In this view, Leibniz's use of infinitesimals is an "*abrégé*" that substitutes a longer classical or Archimedean expression.² This view is correct but limited. It is correct, as we shall examine in more detail below, because Leibniz's methodology does not admit any actual infinite or infinitesimal term. It is however limited insofar as a merely reductionist reading of the infinitesimal to the finite unfortunately obscures the meaning and complexity of the role of the syncategorematical infinitesimals in Leibniz's work.

Reductionist readings of Leibniz's infinitesimal fictionalism are largely made to vindicate Leibniz from the charges of a lack of rigor and, even worse, of contradiction in his admission of infinitesimals in mathematics and his use of infinitary notions in his metaphysics. Part of this vindication begins by noting that, in his early years (pre-1672), Leibniz did entertain a position, inspired by Hobbes, of actual infinitesimals in the form of indivisible parts of motion. Through his mathematical maturation during his *séjour Parisien* (1672 to 1676), he gave up these youthful mistakes and developed the basis for his differential calculus through the notion of a non-actual, syncategorematical infinitesimal. It is in this drive to absolve Leibniz from a youthful position and whatever mistakes his reception might have imputed to him however that we face the danger of over-emphasizing this rejection of actual infinitesimals such as to simply reduce the syncategorematical infinite or infinitesimal into the finite.

I do not wish to suggest that we can read Leibniz's infinitesimals as anything but ideal, syncategorematical or fictional. Simply opposing these characterizations of the infinitesimals to the menace of an actual infinite or infinitesimal however does not allow us much insight into the complexity of Leibniz's reflections in this domain. Leibniz's infinitesimal fictionalism is clearly more complex than what a reductive reading can account for. There are three key places where this reductive reading encounters its limits. First, in Leibniz's metaphysics it is plain to see that there is a dissymmetry between his thesis of a concrete infinite division of the parts of nature³ and his rejection of an actual infinitesimal sum, quantity or magnitude. However, the concrete infinite nature of the world maintains a relation with the admissibility of infinitesimals in mathematics in an indirect way through the principle of continuity that goes beyond the limits of a reductive reading of the infinitesimal. Second, in the domain of mathematics, Leibniz's famous epistolary exchanges with Nieuwentijt over the nature of higher-order derivatives reveals that he affirms these higher-order

differentials, a notion that could be generalized as $dx^n \neq 0$.⁴ As Leibniz himself declares to Nieuwentijt, “Therefore I accept not only infinitely small lines, such as $dx dx$, $dy dy$, $dx dy$. And I accept cubes and other higher powers and products as well, primarily because I have found these useful for reasoning and invention.”⁵ This further development of the structure of these orders of infinitesimal terms at least suggests that a mere reduction of the infinitesimal into the finite is not the only meaning of the qualification of them as “fictional.”⁶ Finally, in the domain of natural science, we can also point to Leibniz’s use of infinitesimal terms in the treatment of actual things in the different writings of his *Dynamics* project. This is seen most clearly in the *Specimen Dynamicum* when he makes a direct association between the measure of dead force and infinitesimal magnitude, albeit with the important caveat that it is a “fictional” one. In none of these cases are we entertaining the notion of an actual infinitesimal, but what is clear is that there are a number of contexts in which Leibniz’s fictional infinitesimals play a role which are not easily explained by a mere reduction of the fictional infinitesimal to finite terms.

By pointing to these three areas of difficulty, I am not proposing to resolve them here. The aim of this article is thus only to provide a critique of this reductionist reading by pointing to two equivocal assumptions that this view is based on. The first equivocation is the association of a foundation of infinitesimals with their ontological or meta-mathematical status. Here, I take up Hide Ishiguro’s important evaluation of the syncategorematical infinitesimal and provide a critique of her reductive reading by showing the ultimate emptiness of these claims. I then provide a short explanation for the historical reasons for this equivocation that is tied to a general equivocation in mathematics and the philosophy of mathematics in the 20th century. The second equivocal assumption that I will critique is the association of the admissibility of infinitesimal terms in mathematics and the rigor of demonstration. I will examine the separability of these two constraints facing Leibniz’s mathematical work in an evaluation of his *De quadratura arithemetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* (1675).⁷ Through an examination of these forms of equivocation at work in a reductive understanding of Leibniz’s infinitesimal fictionalism, I hope to promote a more dynamic view of Leibniz’s relationship with fictional infinitesimals; one that should be sought in a range of different contexts only one of which is his rejection of actual infinity and infinitesimals.

Infinitesimal Fictionalism in The Context of Self-Justification

Although the initial developments of Leibniz’s differential calculus were already developed in his period of maturation in his *séjour Parisien*, notably the *Quadratura*, it was not until much later that this project took an explicit public form in *Nova Methodus pro maximis et minimis, itemque tangentibus, et singulare pro illis calculi genus* (1684) and *De Geometria Recondita et analysi indivisibilium atque infinitorum* (1686). These public expositions elicited opponents and admirers alike and Leibniz became involved in a series of epistolary exchanges (with Cluver, Nieuwentijt) and public debates around the French Royal Academy of Sciences, the very institution for which he originally prepared his *Quadratura* (in 1675) as a means to gain membership.⁸ We

should then understand these debates as an episode of *justification* for the differential calculus. They are then different in nature to both his initial turn to the terminology of the “fictional” infinitesimal and the actual development of the calculus itself.

It is in this later context that we find Leibniz using the expression “well-founded fiction” as a clarification of his own position in a battle where he was not directly a combatant. Leibniz communicated to P. Varignon in 20 June 1702:

Between you and me, I think Fontenelle [...] was joking when he said he would derive metaphysical elements from our calculus. To tell the truth, I myself am far from convinced that our infinites and infinitesimals should be considered as anything other than ideals, or well-founded fictions.⁹

From far away Hanover, Leibniz was asked to clarify questions from a heated debate that was going on in Paris sparked by the first textbook on the calculus, G. de L'Hôpital's *Analyse des infiniment petits* (1696). The wider accessibility of the calculus led to a debate between the proponents of the differential calculus (L'Hôpital, Varignon, Fontenelle) and the opponents (Rolle, la Hire, Gallois).¹⁰ Most notably, M. Rolle's *Un nouveau système de l'infini* (1703) took L'Hôpital's book as a target and laid down a series of attacks that would forecast the eventual attacks of infinitesimal analysis made by Berkeley a few decades later. Although not a neutral observer, the terms of debate over the calculus were in some sense already out of Leibniz's hands. In this context, Leibniz often played a very conservative role, prudentially maneuvering between defending the method of the differential calculus that had come under significant attack and playing down its wilder implications. In the case of this letter to Varignon, the prudential use of “fiction” was rhetorically directed toward the ways in which he felt his theory was over-extended, in this case by Fontenelle, into domains where it did not belong. While one can certainly sense the deflationary meaning of Leibniz's use of *fiction* here, this is far from being the whole story.

In fact what Leibniz explains in his letters to Varignon provides some reason for understanding a different meaning of this deflationary attitude toward the use of infinitesimals. Although the explicit debate in the French Royal Academy between Varignon and Rolle concerned the admissibility of infinitesimal quantities, magnitudes and the like, Leibniz simply asserted the proper borders of this debate. In his view, some of Varignon's opponents in the French context had simply misunderstood the prudential metaphysical scope of the calculus. In his January 1701 letter to Varignon, Leibniz expressed his belief that someone that he held in high regard like Abbé Gallois only spoke against it because, “Perhaps his opposition comes only from his belief that we have founded our demonstration of this calculus on metaphysical paradoxes which I myself believe that we can very well discard.”¹¹ Leibniz also remarks here on his separate debate with Clüver and Nieuwentijt, marking a distinction between debates surrounding the metaphysical implications or assumptions attributed falsely to the calculus and the debates internal to the mathematical use of fictional infinitesimals. We can summarize Leibniz's position here in a phrase

eventually redacted from his 2 February 1702 letter to Varignon, “[I]t is unnecessary to make mathematical analysis depend on metaphysical controversies or to make sure that there are lines in nature that are infinitely small in a rigorous sense in contrast to our ordinary lines, or as a result that there are lines infinitely greater than our ordinary ones.”¹²

We should be careful, as Leibniz is, in clearly separating the debates concerning the metaphysics of the infinite and the mathematical use of fictional infinite and infinitesimal terms. We should also be careful to understand that the separation of these two domains does not directly resolve the problems within each domain. That is, even if we could have a full grasp of Leibniz’s metaphysics of the infinite and infinitesimal, we should still have much trouble in seeing how Leibniz introduces their use within mathematical practice. As I will continue to argue in the following, it is the insufficient distinction between the two kinds of domains and the confounding of the standards of argumentation required in each domain that leads to confusion.

What Is Unequivocal About Fictional Infinitesimals?

Before looking more closely at the two domains of equivocation below, I want to first clear up what is not equivocal in Leibniz’s treatment of the infinite and infinitesimal. There are a number of elements in Leibniz’s work that can help cement this non-equivocal baseline from which to venture into more complex matters. The definition of the mathematical infinite and infinitesimal as syncategorematic is the most important aspect to be treated here. In Leibniz’s youth, evidenced by *Hypothesis physica nova* (1671), there was a period of experimentation with the notion of real indivisibles understood as infinitesimal magnitudes in the extended continuum of motion. As we have already mentioned, Leibniz’s mathematical transformation in Paris from 1672 to 1676 changed all that. Early in this period, and before his work on the calculus, Leibniz had already rejected actual infinite measure. In his notes on Galileo’s *Discourses and Mathematical Demonstrations Relating to the Two New Sciences* in 1672, Leibniz drew the following conclusion from Galileo’s insights, one that he thought the latter should have been forced to accept given his commitments. Leibniz remarked, “Infinity itself is nothing, i.e. that it is not one and not a whole.”¹³ In his texts towards the end of his Parisian period, Leibniz began to employ notions like “fiction” to describe his infinitesimals. This terminological turn coincides with a key conceptual turn towards the notion of a syncategorematic infinitesimal. Although we can attribute this to Leibniz since 1675, for reasons of clarity, we can turn to his elaborated use of this specific notion of the infinite and infinitesimal in his *New Essays on Human Understanding* (1703), his lengthy response to Locke’s *Essay Concerning Human Understanding*.

In his section “On the infinite” in the *New Essays*, Leibniz aims to sidestep a direct confrontation with Locke on the relation of the infinite and its relation with sensible qualities or intuitions. Instead, Leibniz focuses on the idea of the infinite itself. In this rich debate, I move directly to what Leibniz saw as Locke’s mistake. As Leibniz saw it, Locke made the mistake of understanding the finite and infinite as

modes of quantity. Arguing from the perspective of the very concept of the infinite, Leibniz puts it very clearly in the start of his commentary of Locke's "infinity" that,

Properly speaking, it is true that there are an infinity of things, that is to say that there are always more than can be assigned. But, there is nothing as an infinite number nor a line or other infinite quantity, if we take them for true totalities, as it is easy to show. The schools wanted, due to this, to admit a syncategorematic infinite, as they say, and not a categorematic infinite. The true infinite, in rigor, is nothing but in the absolute, that before any composition, and is not formed by the addition of parts.¹⁴

Leibniz's argument here moves away from thinking the finite and infinite as modes of quantity by laying out an important conceptual distinction between three senses of the infinite that he mobilized to help clarify the framework of the debate: the categorematic, the syncategorematic and the hypercategorematic. Briefly put, the categorematic infinite is the infinite as a number, a unity or mathematical entity. This categorematic infinite is contradictory once it is posed, since there will always be a number that can be added to produce a greater sum. The syncategorematic infinite is an infinite that posits that, insofar as there is no greatest number, there will always be a greater number than any given finite quantity. For precision's sake, I will draw from R. Arthur's description that, "To assert an infinity of parts syncategorematically is to say that for any finite number x that you choose to number the parts, there is a number of parts y greater than this: $(\forall x)(\exists y)Fx \rightarrow y > x$, with $Fx = x$ is finite, and x and y numbers."¹⁵ There is finally a hypercategorematic infinite that is invoked in this passage, a "true infinite" which is "nothing but in the absolute." This is the infinite in its absolute sense, the infinity of God or an infinity without parts. As Leibniz suggests in the above, the syncategorematic sits somewhere between the false infinity of the categorematic and the "true" and absolute infinite that is without parts. Now, since Leibniz rejects both the categorematic infinite as contradictory and the infinity of repetition as impossible (or indeed contradictory), the syncategorematic infinite must however also be distinguished from the absolute, that is, hypercategorematic infinite. From this we see that this three-fold distinction is important precisely because it allows us to prevent the development of the syncategorematic infinite from falling into two types of error. The first error is to explain the syncategorematic infinite by means of a "fictionalization" of the categorematic infinite, to make the mediated and non-absolute identical to the immediate and the absolute, that is, to model Leibniz's infinite on the fictionalization of a contradiction. This is no way to read Leibniz's understanding of "fiction." The second error is to explain the syncategorematic infinite by means of a reduction to a "potential" infinite by means of division or repetition. Since what the syncategorematic infinite aims to express – distinguishing itself from the contradictory categorematic infinite and the absolute hypercategorematic infinite – is an actuality that, despite being derived from an origin in the absolute, it cannot simply be reduced to other terms.¹⁶ Without

investigating all the details of the syncategorematic infinite here, we are nonetheless in a position to understand the specific determination that Leibniz gave to the notion of the infinite and infinitesimal.

To be clear, what I am not arguing here is that there is an equivocation regarding these terms. Rather what is equivocal concerns how we interpret their role in foundational matters in Leibniz's mathematics especially with regard to the "fictional" nature of infinitesimals. In order to grasp these problems in a clearer way, my intention here is to provide a non-equivocal basis from which to move into greater complexity.

Mathematical Ontology and the Reducibility of Infinitesimals

In recent Anglophone reception of Leibniz's mathematics, it was Hide Ishiguro's work that has provided the critical push toward a focus on the syncategorematic status of Leibniz's infinitesimals. However, it is also with Ishiguro that we find a strong drive towards a reductive reading of their syncategorematic status as a general reduction of the infinitesimal to mathematical objects as such.¹⁷ What this reading assumes, as I will argue below, is a reduction of the admissibility of infinitesimal terms in mathematics to what in modern terms we might designate as a meta-mathematical or ontological claim about the status of these infinitesimal terms.¹⁸ Although not erroneous, Ishiguro's approach, with the exception of a clear rejection of actual infinitesimals in Leibniz, leads to a number of empty claims about Leibniz's infinitesimals and lends very little to understanding the meaning of the "fictional," status of these syncategorematic infinitesimals.

Ishiguro's reading of the status of infinite and infinitesimals in the second edition of *Leibniz's Philosophy of Logic and Language* provides a rather important example of how Leibniz's fictions are interpreted through a reductive position. The goal of Ishiguro in this part of her book is to vindicate Leibniz from charges of unrigorousness by demonstrating the consistency of Leibniz's notion of the infinitesimals. Ishiguro's text is historically important in her emphasis on the syncategorematic nature of Leibniz's infinitesimals. Her understanding of these terms as well as her general conclusion is highly problematic however insofar as her reductive position erases much of what is important about Leibniz's fictional treatment of these terms.

In Ishiguro's treatment of Leibniz's infinitesimals, she begins her discussion by referring to Abraham Robinson's 1966 *Non-Standard Analysis*.¹⁹ Robinson's project for a notion of a non-finite, non-standard, that is, non-Archimedean concept of an infinitesimal quantity, represents perhaps the most far-reaching attempt to combine the elements of the Newton-Leibniz calculus, the transfinite revolution of Cantor and the formalist methods of the model theoretic theory of sets: the development of a strong concept of the infinitesimal as, formally speaking, a number less than the absolute value of any standard real number ($n < |R|$ where n is a number and R is a standard real number), a project that has, in turn, revived new reflections on Leibniz's legacy.²⁰ Demonstrating the awareness of his participation in a historical lineage that owes much to Leibniz, Robinson's own mathematical treatise devotes a number of

pages to a reading of Leibniz on numbers and infinitesimals and an interpretation of his own contributions along these lines.

Ishiguro's manner of discussing Robinson and the contrast she draws with Leibniz is a clear illustration of the assumptions that drive her reductionist aims in Leibniz interpretation. Robinson posits the infinitesimal as a non-Archimedean (non-standard) *entity*, and the meta-mathematical context of this notion of the infinitesimal qua entity is given through an axiomatic formalism. By contrasting Leibniz to this approach, Ishiguro aims to show that Leibniz's rejection of anything resembling an infinitesimal "entity" leads to a reductive metaphysical position for infinitesimals. What is curious in this intersection between the development of mathematical concepts driven by Robinson's work and Ishiguro's reinvestigation of Leibniz's philosophical-mathematical legacy is that there is an ironic misunderstanding at work. Robinson's contribution, according to his own view, is the development of the concept of infinitesimals in the Leibnizian fashion that had been superseded in the early nineteenth century by Cauchy's re-foundation of the interpretation of infinitesimals in the differential calculus by means of limits rather than as entities.²¹ Whereas the (mistaken) implication of the infinitesimal as an entity had always cast a shadow on the entire method, the notion of the limit has historically proved to provoke less skepticism.²² In turn Robinson, employing a set of mathematical methods made standard by the Cantorian revolution and axiomatic formalism, returns to the original project of Leibniz as interpreted by L'Hôpital and provides a rigorous method of laying out the infinitesimal as entity (and not merely as limit) in his "non-standard" approach to arithmetic, topology and analysis. In fact, the recognition of a distinction between a path of the limit and that of the entity in the development of the infinitesimal concept is at the center of Robinson's emphasis of the originality of his own contribution.

Robinson is quick to note the stark difference between the modern debates around the metaphysics of mathematics and that of earlier periods. Insofar as Cantor and the mathematicians of the *Grundlagenkrise der Mathematik* had focused on the globally logical character of the issue of mathematical objectivity, the reality and objectivity of mathematical entities and their surrounding methodology are treated as a whole and not separated between "normal" mathematical entities like the natural numbers (1, 2, 3...) and numbers with special status like Π , i , irrational numbers and even infinities and infinitesimals. Robinson remarks that,

[W]hatever our outlook and in spite of Leibniz' position, it appears to us today that the infinitely small and infinitely large numbers of a non-standard model of Analysis are neither more nor less real than, for example, standard irrational numbers. [...] For all measurements are recorded in terms of integers or rational numbers, and if our theoretical framework goes beyond these there is no compelling reason why we should stay within an Archimedean number system.²³

After making this claim, Robinson deftly turns to Leibniz's 1701 letter to Pinson, cited above, "On ne differe du style d'Archimède que dans les expressions [...]," implying that even if the non-standard infinitesimals go beyond standard expressions, they are in not in conflict with them precisely in the sense that they do not cover the same field of application.²⁴ What he then makes clear regarding his relation to Leibniz is that the debate of the status of "his" infinitesimals should, on the one hand, be held in common with the debate on the status of real numbers and indeed number as such.

Robinson's remarks here demonstrate his clarity on two important points. First, Robinson understands that Leibniz's own determination of the mathematical infinite and infinitesimal was a syncategorematic one and therefore Archimedean in nature. Secondly, he understood that his own notion of the infinitesimals are to be interpreted systematically and holistically within modern meta-mathematical and mathematical standards and hence at a certain distance from 17th century context of rigor and admissibility. As he continues, he touches on Lagrange, d'Alembert and finally Cauchy's deepening of the calculus project by means of limits. In this, though Cantor himself rejected the possibility of infinitesimals qua number, the Cantorian revolution of method for tackling these problems provided the grounds for Robinson's own infinitesimal quantities. Robinson uses the reference to Leibniz in order to establish a historical understanding of some of the continuities, but also, more importantly, to mark the discontinuities that exist between his project and its Leibnizian inheritance.

Ishiguro distances her reading of Leibniz from Robinson by making a clear distinction of camp. In her view, Robinson, by the standards of modern mathematics and that of Fregean and Russellian contextual reference to mathematical objects, successfully argues for non-Archimedean infinitesimals. Yet even if we accept the results of his work, "his belief that, even if Leibniz considered infinitesimals to be an ideal fiction which can be dispensed with, these infinitesimals for Leibniz were fixed entities with non-Archimedean magnitudes, the introduction of which shortens proofs, seems unwarranted."²⁵ We should note that this is a gross misreading of Robinson, for whom Leibniz served only as a forebear of his method, as the latest development of the historical path of the infinitesimal that privileged an entity-like conception. Nonetheless, what Ishiguro hopes to argue is that a proper reading of the notion of the infinitesimal puts us in a theoretical continuity with an alternative historical path that privileged a limit-like conception. The main reasoning for this is that a proper understanding of Leibniz's syncategorematic infinitesimals will place it in continuity with traditional or standard Archimedean sums that follow the principle of the whole being greater than a part.

Ishiguro's argument then begins with an *incorrect* construal of Robinson's view of Leibniz where infinitesimals are fixed, categorematic entities. She then argues against this by demonstrating their syncategorematic status in Leibniz's work. From here, she proceeds to show that this implies that Leibniz's treatment of syncategorematic infinitesimals is effectively what gives the entity character of infinitesimal magnitudes a fictional status. This fiction does not allow for any true

entity but rather, through Leibniz's use of mathematical symbolism, only captures the relationship between two finite but variable points that ultimately obey the Archimedean principle of the whole-part relation.

Despite these stated misgivings in her reading of Robinson, Ishiguro does provide one very pertinent assessment of how one is to understand the syncategorematic character of infinitesimals in Leibniz. First, the distinction that she employs, the distinction between entity and limit, is based on the identification of real or actual infinitesimals as a special, non-Archimedean type of number, the very type of number that Robinson himself aims to develop in *Non-Standard Analysis*. Her argument relies on the fact that there is no such special status given to the infinitesimal in Leibniz's own work. But in Leibniz's own words, as Ishiguro is keen to point out, "This is why I believed that in order to avoid subtleties and to make my reasoning clear to everyone, it would suffice here to explain the infinite through the incomparable, that is, to think of quantities incomparably greater or smaller than ours."²⁶

Here, it is only apparent that Leibniz leaned toward just the kind of incomparability that would give infinitesimals this "incomparable," that is, non-standard character. It is precisely this sort of claim that would have given rise to ambiguous interpretations of infinitesimals even in Leibniz's own day. As Ishiguro points out, however, there is no great mystery about how one is to understand these magnitudes or quantities termed "incomparable," since the relation between the side of a plane to its area (a one dimensional comparison with a two dimensional) is the standard sort of relationship referenced by the incomparable. The term incomparable distinguishes two quantities and two dimensions but does not give to any of these, taken in themselves, a non-Archimedean status. The incomparable is thus a *neutral* term in this debate about the status of the infinite and infinitesimal.²⁷ As Ishiguro continues to argue, the supposedly infinitesimal magnitude in question, the differential dy/dx is not infinitesimal but rather a finite relation between differentials (unless dy and dx are infinitesimals of different orders).²⁸ Here, the incomparability in question concerns the relationship between the magnitudes themselves and not, as it might seem, the relation between finite and infinitesimal magnitudes. Though she does not give an example, this neutral notion of the relation of incomparability can be said of the relation between the side of a polygon and its area.²⁹

From this clarification of the equivocation of the use of (in)comparability, Ishiguro's argument turns to introduce the syncategorematic character of the infinitesimal to "save" Leibniz from misreadings that identify "incomparable" with the "non-standard." She quotes Leibniz's argument to Varignon where he argued for the non-rigorous identification of rest as a kind of motion and a circle as a kind of polygon by virtue of the fact that,

[R]est, equality and the circle terminate the motions, inequalities and the regular polygons which arrive at them by a continuous change and vanish in them. And although these terminating points [*terminaisons*] are excluded, that is, are not included in any rigorous

sense in the variables which they limit, they nevertheless have the same properties as if they were included in the series, in accordance with the language of infinites and infinitesimals, which take the circle, for example as a regular polygon with an infinite number of sides.³⁰

From this, Ishiguro argues that here we find Leibniz describing the infinite in many of the same ways that contemporary students of calculus would, that is, with the terminating points treated as limits that are approached by a function (like a regular polygon approaching circularity by virtue of increasing its number of sides) without rigorously taking the last point into the function itself. She adds to this, however, a caution:

I think Leibniz is misleading when he writes to Varignon that truthfully speaking he himself is not sure whether he shouldn't treat infinitesimals as ideal things or as well-founded fictions. The limit may be a well-founded fiction, but talk of infinitesimals is, as he says, syncategorematic and is actually about 'quantities that one takes [...] as small as is necessary in order that the error should be smaller than the given error.'³¹

According to the idea of the infinitesimal's syncategorematic status, Ishiguro's interpretation of Leibniz naturally leads to favor the reading of Leibniz's infinitesimals as a proto-limit, that is, a proto-Cauchyian idea. In this, she strongly identifies the syncategorematic understanding of the infinite and infinitesimals with the limit concept. Here Ishiguro argues that the combination of the infinitesimal with the notion of the limit allows us, as it does in common usage today, to speak of the variables on a function approaching zero as a limit. As such, Ishiguro remarks:

[W]e have seen that Leibniz denied that infinitesimals were fixed magnitudes, and claimed that we were asserting the existence of variable finite magnitudes... we could say that Cauchy claimed that limits existed whereas Leibniz wanted to say that they were a well-founded fiction.³²

According to this view, the difference between Leibniz and Cauchy would then be "very little."³³

This interpretation of Leibniz's infinitesimal as a proto-limit is accurate in many respects. In retrospect since Leibniz did not commit to an actual infinite or infinitesimal entity but proposed a syncategorematic one, Ishiguro's proto-limit interpretation of Leibniz is compelling. However, she is only half right. That is, the larger aspect of her argument seems to skew the many different attempts to both put the idea into question in Leibniz's own reflections and the different ways in which the attempts to defend his conception provide opportunities for him to reinvent or make nuances in his use and significance of the term. An obvious mark of this is how

Ishiguro's argument turns on the false dichotomy between a Robinsonian, non-Archimedean or non-standard "entity" nature of the infinitesimal, and a proto-Cauchyian limit or finite variable nature of the infinitesimal. Indeed Leibniz did not view this alternation between a fictional entity and a limit as a problem for developing a new class or type wherein these quantities exist. Leibniz simply did not involve himself with ontological questions of this sort.³⁴ In turn, Leibniz's problem seems to have been how one is to understand the status of infinitesimals given the necessary constraints that they are not in any manner fixed quantities, unities, wholes or numbers.

Ishiguro argues that in order to understand the status of infinitesimals in Leibniz, we should not hesitate to import Frege's idea of contextual theory of meaning and Russell's idea of definite descriptions, two fundamental principles of the contemporary philosophical approach to mathematical objectivity. For Ishiguro, the Fregean idea of *salva veritate* applies entirely to Leibniz's use of infinitesimals in calculation, and thus there should be nothing, in terms of the conservation of truth values, to mark Leibniz's use of the infinitesimal as inconsistent with contemporary evaluations of mathematical propositions. In this, she also notes that Frege explicitly remarks on the differential, "It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts and also their content. This observation is destined, I believe, to throw light on quite a number of difficult concepts, among them that of the infinitesimal."³⁵ On the other, Russellian side, she remarks that as a special example in his section on definite descriptions in the *Principia Mathematica*, "Russell [...] gave the sign for the differential d/dx as his first example of a sign with which we are familiar which has a meaning only contextually."³⁶ Noting that Russell's theory of definite descriptions is aimed at unique existentials, she invokes the syncategorematic status of infinitesimals as that which determines a limit. This determination, through the use of a symbolic notation dy/dx , puts the Leibnizian determination of the infinitesimal in consistency with the modern, that is, post-Russellian view of the logical status of mathematical objects. In the end she argues, despite the differences between Frege and Russell, Leibniz's views on infinitesimals can be consistent with both theories.³⁷

This reductionist reading, although apparently metaphysically neutral, relies on a rather heavy metaphysical assumptions about logical consistency, successful reference and objectivity. More importantly, in construing the problem of infinitesimals in this manner, however, we remove the need to ask what makes infinitesimals "fictional" with respect to other references from number. That is, in the attempt to vindicate Leibniz's infinitesimal from a Russellian misunderstanding, we end up losing what is specifically Leibnizian in the approach to this problem.

This confusion allows us to see the ultimate irony in Ishiguro's position. The fact is that Robinson, as Ishiguro also claims, had already laid the foundation for a rigorous, albeit non-standard use of infinitesimals qua entity. Thus Robinson's infinitesimals, according to the standards of modern, post-Cantorian means of argumentation, fulfill both the Fregean and Russellian conditions of mathematical legitimacy raised in her evaluation of Leibnizian infinitesimals. Ironically, her

evaluation seems to be lacking in the very places where Leibniz's views would require the most explanation, in the deployment of infinite and infinitesimal terms as specifically Archimedean. In this sense, her justification of Leibniz's infinitesimal suffers from both a historical treatment as well as an attention to the specific ontological question lurking behind it. The problem I refer to here lies in that the question that Ishiguro attempts to resolve is neither Leibniz's nor Robinson's. Indeed, Leibniz's difficulty with the status of infinitesimals lies not in his already formed idea of a fundamental criterion of the exclusion of non-Archimedean wholes, but rather in its conceptual coordination with Archimedean terms. In this sense, even if we appreciate her focus on the syncategorematic status of Leibniz's infinitesimals, her reductive reading is one that does not clarify the meaning of these syncategorematic infinitesimals that she has so carefully emphasized. In turn, she associates their well-foundedness with contextual reference, a move that anachronistically removes us from a position of being able to treat their meaning and status qua fictional and syncategorematic in distinction from other more standard sorts of numbers.³⁸

A final important remark concerning this form of equivocation has to do with the role that Leibniz has played in the recent history of mathematics and the philosophy of mathematics. The nature of Leibniz's reception today, especially its Anglophone context inaugurated by B. Russell himself, has a lot to do with the 20th century confrontation of the *Grundlagenkrise* that found deep inspiration in Leibniz's grappling with the continuum, logical predication, binary mathematics (through computing) and the vision of a *mathesis universalis*. Of course, there is no denying that visionary thinkers like Cantor, Frege, Russell, Gödel and others drew inspiration from Leibniz. What becomes problematic is how Leibniz's interpretation becomes indexed, retroactively, to the standards of this *Grundlagenkrise*.

The result is that questions asked of Leibniz become immediately those questions asked by those thinkers of the *Grundlagenkrise*; taking for granted the convergence of the question of the ontological status of mathematical objects and the methodology through which mathematics is grounded. This is most obvious in the work of Frege and Russell where the method through which mathematics is grounded is simply a theory of mathematical ontology, an identification that will eventually be *philosophically* critiqued in Benacerraf's famous 1965 article, "What numbers could not be."³⁹

The closing of the *Grundlagenkrise*, due in part to Gödel's and Cohen's results, does not mean the end of foundational questions in mathematics, but only its relativization. That is, a number of different accounts of mathematical foundations can be treated simultaneously without the requirement of homogeneity. This in fact demonstrates that the temporary unification of questions about foundational ontology and the foundations of the methods in mathematics was one that was historically stitched together in an epoch where logicism provided the main challenge. In the later 20th century, the new wave of category theory, as the main pretender to the throne of foundation, has been capable of specifying in what way category theory may or may not be a foundational account of mathematics. Jean-Pierre Marquis, an avid supporter of the importance of category theory, has laid out not less than six different senses

(logical, cognitive, epistemological, semantical, ontological, methodological /pragmatic) of a “foundation” for mathematics.⁴⁰ What Marquis emphasizes is that a foundational account for mathematics is now relativized along the lines of a pragmatic (or methodological) set of “toolbox” notions more sensitive to the methodology of the “working mathematician” than to the question of an ultimate logical status. With this, it is not so much that we stop asking ontological questions about mathematical terms, objects and operations. It is rather that these questions no longer seek a neat and common solution.

The Admissibility of Infinitesimals and the Rigor of Demonstrations

My critique of Ishiguro’s anachronistic approach that leads to equivocation seems to immediately recommend a “toolbox” approach, borrowing Marquis’ terms, for dealing with Leibniz’s fictional infinitesimals; we shall see below how this approach can also lead to equivocal results as well. Here, I argue again that a reductionist approach to infinitesimals clouds our capacity to understand the foundational constraints through which Leibniz developed his mathematical methods. What is equivocal in this case concerns the “order of reasons” in understanding the rigor of Leibniz’s infinitesimal methods. As I shall argue, the problems of the admissibility are staked on a deeper and more historically entrenched problem of the form of proof. It is in addressing this “thorny” problem of the form of proof that Leibniz introduces his infinitesimal fictions. What we should understand is that the reducibility of Leibniz’s infinitesimal methods into finite terms is not the fulcrum upon which the rigor of his method is balanced. Rather, the priority of the problem of proof renders the admissibility of infinitesimal methods as a secondary problem. Here the equivocation that I aim to critique lies in the direct association of rigor in Leibniz’s methods with the reducibility of infinitesimal terms. This equivocation both occludes the problem of rigor in mathematics as Leibniz understood it and overshadows the complexity of infinitesimals in Leibniz’s thought.

My choice for a concrete look at Leibniz’s methodology is the *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* (1675). This rich and complex text represents one of the jewels that have been delivered by the efforts to carefully organize Leibniz’s mathematical corpus in the Akademie edition. First edited by E. Knobloch, to whom we owe this new-found accessibility, the text represents one of Leibniz’s rare and longest purely mathematical treatises written in such a format. It is a highly polished text, conceived by Leibniz as a means of entering the Royal Academy of Sciences, in Paris; it is now available in a bilingual edition (Latin text by Knobloch and French translation by Parmentier). A good reason for looking at this text is also its distance from the later justificatory treatments of infinitesimals where the questions to which Leibniz was made to respond to were not exactly his own, but framed around debates in the French context. By presenting the constraints that Leibniz set down for himself in the initial development of the methods of the differential calculus, these texts shine a new light on Leibniz’s thinking.

The *Quadratura* is too rich and complex of a text to examine here in a comprehensive way. Yet I wish to highlight a number of its features salient to our discussion, and this cannot be done without taking a quick look at a fragment of the foundational portions of its argumentation. In this text, Leibniz's ambition was to treat not only the "geometric" curves of apollonian conic shapes (the circle, ellipse, hyperbola, etc.), but rather to open up a possibility for understanding the complex "mechanical," that is, logarithmic, curves. The ambition of this text on Leibniz's part is plain to see, but we shall confine ourselves to the rudimentary development, in the first 7 propositions, of the basic method.⁴¹

The first five propositions are preparation for the demonstration of a quadrature of the half circle starting with the sixth proposition and determination of the area of a triangle in proportion to a triangle determined by the height of said triangle in line with Euclidian geometry and in defining a series number of arithmetic relations. The goal, starting at the sixth proposition, which Leibniz notes as "most thorny" or "*spinosissima*,"⁴² is to provide an argument for the quadrature of a quarter-circle through the equation of the area under the half circle with the added sum of a series of rectilinear figures, in this case, a series of obtuse triangles. At the start of the sixth proposition Leibniz begins with a quarter-circle C between two axes, A the horizontal axis and B the vertical axis. He then chooses a number of points on C, 1C, 2C, 3C, 4C, etc. This series of points on C are thus defined as a point between a series of abscissa and ordinates, 1A1B for point 1C and so on. Taking the tangents of these points on C, he finds the intersection between the tangents and the horizontal axis A to define a series of points 1T, 2T, 3T, 4T, etc. From this, Leibniz produces a second series of points 1D, 2D, 3D, 4D, etc. by defining 1D as 1T1B, 2D as 2T2B, etc. The series of new points gives us a new curve D, defined by the tangent lines on the original half circle curve C and the ordinates of points on C. In defining this new curve D, Leibniz will bracket the quadrature of the original curve C for the moment in order to provide a quadrature of this auxiliary curve.

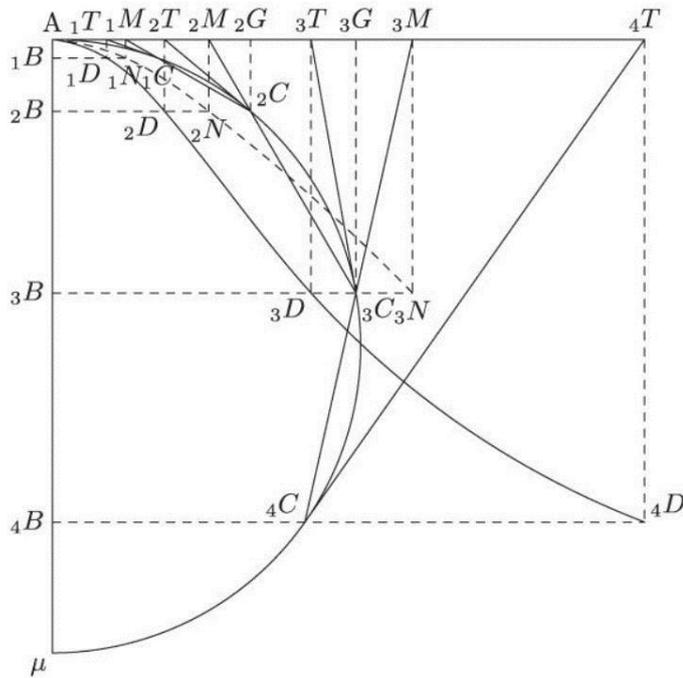


Fig. 1⁴³

After defining the auxiliary curve D through the points 1D, 2D, 3D, etc. Leibniz defines a series of points by taking the chords between the original curve C: between 1C and 2C, between 2C and 3C, between 3C and 4C, etc. The chords will intersect the horizontal axis A and give a new set of points 1M, 2M, 3M, 4M and the like. Taking once again the series of points M (as abscissa) and the points B (as ordinates), Leibniz creates what he calls a “*spatium rectilineum gradiforme*” (shaded area in the figure below) or what we might call a “step function.” In short, this designates a rectilinear figure G whose area is comparable to the area under the curve D.

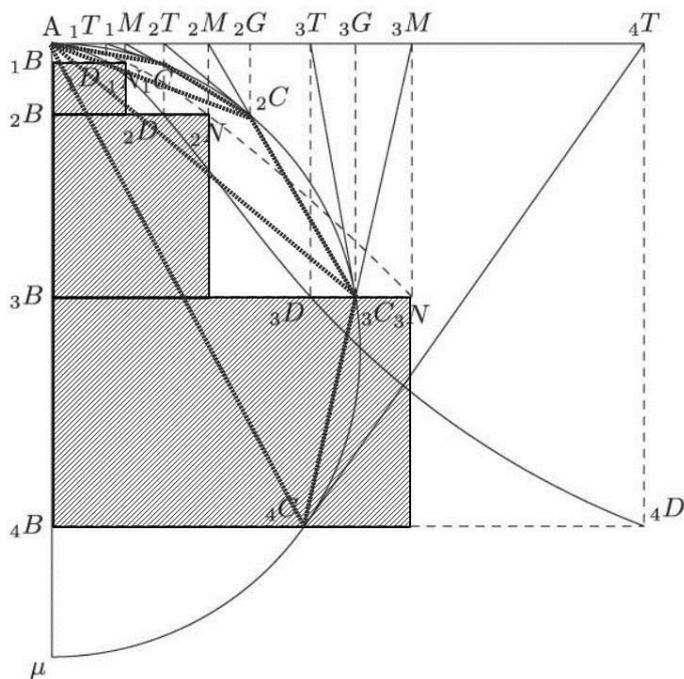


Fig 2⁴⁴

In brief, Leibniz's argument concerning the auxiliary curve D and the rectilinear figure G consists in showing that by placing the points C more closely together, we can reduce the difference between the areas under the auxiliary curve D and figure G. Here we can simply notice that at any given interval, there is a difference between the figure G and the curve D but we also see that the more points on C, points closer together, that we start with, the more this difference diminishes as the rectilinear figure becomes more finely graduated. Conforming with the intuition of a circle as an infinitely sided polygon, the intuition in this case is that such a gradiform rectilinear space can become indiscernibly identical with the curve in question.

From this Leibniz moves on to generalize the results about this rectilinear gradiform space G and the auxiliary curve D for the original curve C by using the set of arcs between 1C and 2C, 2C and 3C, etc. that were already involved in his previous reasoning about the auxiliary curve. Hence, starting with the origin A, we have a series of obtuse triangles A1C2C, A2C3C, etc. that are inscribed in the half circle C (see figure 2). Leibniz's argument here aims to identify the area of this series of triangles with the half circle C. The argument that he gives proceeds by reduction *ad absurdum*. Leibniz was careful to note in the first proposition of the *Quadratura* that the relation between a triangle and a rectangle of the same height and length is that of 1 to 2. Here, we see that these inscribed triangles correspond to the rectangles in the gradiform space in terms of height and length since both are determined by abscissa

determined by the chords drawn on C and the ordinates of $1C2C3C$, etc. points on C. As such, the area of the combined triangles T should be exactly half that of the combined rectangles Q.

From this constructive basis, Leibniz then moves to a demonstration for an accomplished quadrature. This takes place in the form of a *reductio* proof. For purposes of *reductio*, we pose that there is a difference between these two figures and that there is a difference Z between Q and twice T.

$$Q - 2T = Z.$$

Following the same reasoning, we can imagine a polygon P inscribed inside figure T which would allow us to say that the difference between P and T would be less than the difference between the quarter of Z.

$$T - P < Z/4.$$

In the same way, we imagine a new gradiform space G in Q. In the same way, we obtain that the difference between Q and G and the quarter of Z.

$$Q - G < Z/4$$

We then apply these two inequalities to the original formula asserted for purpose of *reductio*. We obtain:

$$Q - 2P < Z/4$$

$$2P - 2T < 2Z/4$$

By the principle of triangular inequality treated by Leibniz in proposition 4, we obtain:

$$Z = Q - 2T < |Q - 2P| + |2P - 2T| = 3Z/4$$

Quod est absurdum. As Leibniz argues, “It is thus impossible to suppose any difference since Z was undetermined and could have represented any [quantity]. Twice the trilinear [figure] is thus equal to the quadrilinear [figure].”⁴⁵ This argument results in contradiction and we arrive at the conclusion that there is no difference between Q and 2T.

There are two aspects of Leibniz’s argument here that we should linger over. As other commentators have underlined, what we can gather from Leibniz’s methodology here is that it is clearly classical in its presentation and manner of proceeding. In Knobloch’s evaluation, he insists that this classical mode does render novel methods which puts this text squarely in the modern, that is, post-Greek, context.⁴⁶ This modernity is represented by the second, more delicate, aspect of the proof, Leibniz’s inventive use of the *reductio* argument.

For the first aspect, it is clear that Leibniz sought to base his demonstrations on a solid foundation of construction. This methodological ethos on Leibniz’s part required, as we saw, a longer demonstration that required the creation of an auxiliary curve and a grounding in triangular relations à la Euclid. Here, as many commentators will mention, Leibniz plays on an intuition of “passing to the limit” and comes very close to the ϵ - δ definition of limits that we are familiar to today due to the work of mathematicians from Cauchy to Bolzano.⁴⁷ Despite this, Leibniz proceeds in the sort of rigor that connected his argument with an Euclidian and Archimedean methodology which may have even appeared conservative according to the spirit of the times.

The second aspect, setting Leibniz apart as truly modern, concerns the indirect or *reductio* argument utilized to deliver a final synthesis on the initial layer of constructions in his *Quadratura*. This does indeed deliver something of a modern flavor. Yet what is modern is not exactly Leibniz's use of the *reductio* form that was commonplace in classical attempts at quadrature through exhaustion (Euclid and Archimedes). What is modern in Leibniz's argument here is how he uses this *reductio* form in the course of his demonstration and in relation to his constructions. As it turns out, this is also where Leibniz makes explicit mention of the "fiction" of infinitesimals. Leibniz himself explains in the scholium to proposition 7,

I admit of never having heard of a method capable of perfectly demonstrating the quadrature that is not an *ad absurdum* quadrature; I even have reasons to worry that we might not do so in a natural manner without introducing fictive quantities, I mean infinites or infinitely small; however among all the deductions *ad absurdum*, I think that the most simple, the most natural manner of proceeding and the closest manner to a direct demonstration consists in directly showing (without which we are ordinarily led to a double reasoning in proving that one is not, on the one hand greater and on the other hand less than the other) that there is no difference between two quantities and these quantities are equal.⁴⁸

As we see here, what is truly innovative about Leibniz's approach, as he was certainly aware, is that he provides a "more simple" and "more natural" approach that is at a certain distance from the traditional mode of a double *reductio* that is needed in the strictly Archimedean approach. The significance of this Leibnizian innovation is precisely grasped in S. Levey's evaluation of the argument,

No direct construction of the area of the quadrilateral by means of a single step space would be possible without representing the step space as composed of infinitely many infinitely small (narrow) rectangles. But with the new principle of equality in play, it suffices to show that any given claim of finite inequality between the two areas can be proved false by some particular finite construction, even if there is no single finite construction that at once gives the quadrature of the curve exactly. No 'ultimate construction' lying at the limit is required. Under the aegis of the principle of equality, the system of relations among the series of finite constructions already proves the equality; Leibniz's novel technique of elementary and complementary rectangles thus obviates the need to appeal to infinitely small quantities altogether.⁴⁹

In looking at the *Quadratura*, we can easily agree with Levey's emphasis on the lack of any appeal to anything other than finite terms. Of course, as Levey also notes,

this result is made possible by a “new” principle of equality and hence the argument in *Quadratura* is not merely a repetition of Archimedean exhaustion but constitutes an ingenious advancement. In this advancement, Levey continues to argue,

It goes without saying that his technical accomplishments in quadratures far outstrip the original reaches of the method of exhaustion... [A]t the level of the basic logic of the proof strategy, Leibniz’s reasoning in Prop. 6 very much bears the stamp of Archimedes; perhaps we should call it a neo-Archimedean style of proof.⁵⁰

Leibniz’s neo-classicism is clear not only in the mathematical domain but also in his thought in general. His appeal to Archimedes in mathematics is powerfully mirrored by his appeal to Aristotle in metaphysics. Yet, just as Leibniz’s metaphysics is not simply reduced to Aristotle, his mathematics represents a significant leap from its Archimedean roots. With one foot in the past and one foot in the future, Leibniz’s often remarked eclecticism is an emanation of this theoretical attitude that is at once conservative and progressive.⁵¹

In examining Leibniz’s method in the *Quadratura*, we can see how Leibniz’s infinitesimal fictions are to be considered as reducible to finite terms. This directly corresponds to the syncategorematical infinite that Leibniz can be shown to have unequivocally held at least since the mid-1670’s and anticipates the ϵ - δ definition of limit that will eventually be established through Cauchy and Weierstrass. The first aspect of Leibniz’s argumentation highlighted above shows his intent to remain in the classical constructive tradition. The second, innovative and modern, aspect of Leibniz’s argumentation lead us to understand that the progressive dimension of his work, seen in his use of the *reductio* argument, is made in view of this first aspect. In both of these aspects, we understand how Leibniz’s use of infinitesimal fictions amounts, in practice, to reasoning with finite terms in a way that represented an innovative leap from classical mathematical practice. In this sense, we can strictly maintain that the interpretation of Leibniz’s infinitesimal fiction as methodologically reducible, qua *abrégé*, to purely or merely finite expressions is legitimate.

Although legitimate, this approach encounters some immediate limits however and we can begin to see this by considering the weight that this methodological reducibility carries in the rigor of the demonstration. Leibniz’s concern with rigor in his argument does not lie with the status of infinitesimals. We can see this directly in the worry that Leibniz expresses in the scholia to proposition 7 quoted above. It was the use of indirect demonstration that bothered Leibniz, but he ultimately affirms that it is through the use of these fictional entities that his argument through inequalities can be understood as “more simple” and “more natural” than earlier approaches. With respect to the standards of rigor, Leibniz is not so much concerned with these fictive entities but rather the form of proof. His manner of proceeding is a departure from standard means of employing indirect or *reductio* demonstration but he remains confident that his version of a *reductio* argument,

through his new principle of equality, is as close as one might get to a direct argument concerning the quadrature. As Mancosu argues in his recent *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, however, much of the questions of rigor in post-classical mathematics leading up to Leibniz time concerned the debate over the preference for direct or a priori proofs over indirect or *reductio* proofs; a debate over rigor that profoundly impacted Cavalieri and Descartes was also one that did not escape Leibniz.

Clarifying the sort of questions that were asked by Leibniz's near contemporaries about rigor can help us resituate the context of Leibniz's argument here. As Mancosu has carefully pointed out, the movement of mathematics from Ancient Greece through the Renaissance toward the modern period should not exactly be traced along the path of the concept of infinity. Here we can add that it is only through the opening up of Cantor's paradise that it might suit us to understand the modern period as eventually leading to transfinitude and axiomatic set theory. Mancosu points to a number of important Greek figures like Archimedes and Theon who already made generous use of exhaustion and indivisibles in quadrature problems.⁵² The 17th century was no doubt a period where these authors and their methods sparked a renewed interest such that we might call this period one of "rediscovery." The key difference is that, as Mancosu writes, "For the Greeks there was no doubt that infinitary techniques, like the ones employed in the indivisibilist method, could not be formally accepted and had only heuristic value."⁵³ In the movement to modernity, through the late Renaissance, mathematical rigor came under the scrutiny of Aristotelian method. Concerning the possibility of geometry's inclusion into Aristotelian science, a debate raged over whether mathematical demonstration can achieve the status of the requirements of the knowledge of causes set out in the *Posterior Analytics*.⁵⁴ Two kinds of demonstrations in the foundational sections of Euclid's *Elements* presented a barrier to geometry's inclusion into Aristotelian science. These are Euclid's frequent use of indirect or apagogic proofs and proofs by superposition. Taking up the problem of apagogic demonstrations, late Renaissance mathematicians faced the problem that these indirect proofs mean that a theorem is demonstrated only in view of the absurdity of its contrary and not in view of its (Aristotelian) cause. Of course not everyone accepted this view and, as Mancosu carefully lays out, these were those who defended geometry on the basis of its direct proofs while qualifying the indirect proofs as heuristic, others defended indirect proofs as causal and still others who argued that the progress of geometry can deliver direct proofs for those results that are only temporarily given as indirect proofs.⁵⁵

The inheritance of this debate still deeply influenced the exchange between Guldin and Cavalieri and it was still the form of demonstration that framed the rise of these debates about infinitesimal or indivisibilist methods. There is thus no surprise that Archimedes served as an important figure of contestation in this context precisely because his method of exhaustion involved both a (double) apagogic demonstration and an appeal to infinitesimals. To rectify this context of the question of mathematical rigor, Mancosu will emphasize that, "What is new in the seventeenth century is the

effort on the part of several first-rate mathematicians to grant these heuristic techniques a formal validity.”⁵⁶

The unification of heuristic infinitesimal techniques, dating back to Archimedes, and formal validity in the ideal of direct proof, under the influence of the standard of Aristotelian science, provided the grounding problems of rigor confronted by 17th century mathematicians including Descartes and Leibniz. This perspective can help us gain some distance from the standard reconstruction of the 17th century as characterized simply by the destiny of Koyré’s “infinite universe.” This perspective also helps cast a proper light on what is indeed “*spinosissima*” about Leibniz’s proof above.

As Mancosu points out in his analysis of *Quadratura*, “Leibniz seems to be involved in a vicious circle. He asserts that the justification for infinitely small or large quantities can only be given using proof by *reductio*; then he claims that only by using infinitely small or large quantities can one give direct proofs.”⁵⁷ In what we have noted earlier, Leibniz’s argument in the *Quadratura* is balanced on the fulcrum of the form of demonstration. According to the ideal of direct demonstration, the finitude of Leibniz’s demonstration does not fully qualify it as rigorous. Although understood as less rigorous than a direct method, Leibniz remained closer to an Archimedean method, or, as Levey puts it, a Neo-Archimedean method. Leibniz can be understood as innovative in this respect on its own rights but we are far from qualifying this argument as “rigorous” within Leibniz’s own context. No doubt, this is precisely why Leibniz considered his demonstration as “thorny.”

In this context what I have emphasized as corresponding to the question of rigor is, as Mancosu has convincingly argued, an epoch defining debate surrounding the form of proof. In this light, we see that Leibniz’s argument was balanced on a fulcrum between methodological rigor and the admission of limit-like infinitesimal fictions. As we have seen, methodological rigor demanded the admission of these infinitesimal fictions. Here, a focus on what was, for Leibniz, the problem of rigor reveals a picture that is quite different from what we might expect. That is, limit-like infinitesimal fictions were introduced for *the sake of rigor* rather than at the cost of rigor. This observation is more capable to rendering apparent what was truly brilliant about Leibniz’s approach to quadrature and his development of the differential calculus. As such, to read the *Quadratura* along the lines of rigor is indeed to underline its brilliant use of indirect proof and his innovative notion of equality rather than that of the uses of infinitesimals. On this same point, Levey quite correctly notes that, “Leibniz’s demonstration of Prop. 6 is ‘rigorous’ in the modern sense of involving only finite quantities; it makes no reference to infinite or infinitely small values.”⁵⁸ The attempt to judge (and vindicate) the rigor of Leibniz’s *Quadratura* along the lines of his prudential use of the infinitesimal qua fictions, reducible to finite terms, is a judgment made from a modern perspective.

What is equivocal in reading the *Quadratura* is the equivocation of what it means for Leibniz to put forth rigor in mathematical method. It turns out that what we seek with the notion of rigor does not correspond to Leibniz’s own concern with rigor. Leibniz’s concern was more wary of the form of demonstration, rather than

being a direct concern about the reality of the infinitesimal. The risk, as Leibniz, saw it, concerned his real innovation in his form of proof and his treatment of the constructive aspect of his demonstration. As such, the emphasis on suggesting that Leibniz does not employ anything other than finite terms in his argument can only be said to conform to a modern and contemporary way of thinking about rigor. This distinction between how the question of mathematical rigor was expressed in Leibniz's time and how it is conceived in our time provides reason to understand what is equivocal in the search for the foundations of Leibniz's infinitesimal fictions.

By highlighting the different senses of the question of rigor between Leibniz and our time, I do not wish to simply efface the problem of the infinitesimal. As other commentators have shown in sufficient measure, Leibniz simply did not engage in the treatment of infinitesimals as actual and hence did not in any case tempt the contradictions concerning the infinite and infinitesimal that had been well established since Zeno's time. Reading the problem of Leibniz's infinitesimal fiction in the light of this emphasis on the distinction between concerns of rigor and metaphysical questions helps us grasp why Leibniz will insist, later in his life, that mathematical method should not depend on metaphysics. The corollary of this assertion is then that mathematical method does not directly entail, except perhaps in providing some conceptual limits, anything about the metaphysics of infinitesimals. Now, we are far from saying that Leibniz still held on to something like an actual indivisible or infinitesimal. Yet looking at the meaning of the fictional infinitesimal from this standpoint does cause some hesitation in accepting a full reduction of fictional infinitesimals qua *abrégé* to merely finite terms. It is only from a retrospective approach that this focus even makes much sense.

Conclusion

As I have mentioned briefly above, Leibniz's short and youthful romance with actual infinitesimals in the form of indivisibles in his considerations of bodies in motion was definitively over around the time of his reading notes on Galileo in 1672. In this, Leibniz's texts after this period showed a sharp terminological change toward a "fictional" infinitesimal and eventually an unequivocal syncategorematic determination of the infinite and infinitesimal. Simply grasping this turn and this determination of the infinite and infinitesimal however does not, as I have argued, shed much light on their meaning and their role in Leibniz's mathematics and metaphysics. The drive to vindicate Leibniz from accusations of holding something like an actual infinite and infinitesimal quantity, magnitude or unit has led to a purely reductive reading of the syncategorematic infinitesimal to the finite. While this judgment is legitimate in some respects, it is limited in its capacity to address a number of different uses of the infinite and infinitesimal (understood as strictly syncategorematic) in his general thought. More problematically however, the focus on this vindication of Leibniz had occluded an appreciation for Leibniz's own reflections on the role and use of fictional infinitesimals. The result is a series of equivocations concerning the foundations and rigor of these infinitesimal fictions.

In my critique of Ishiguro, I have pointed to the error in reducing Leibniz's fictional infinitesimals to a general treatment of mathematical objects as such. The qualification of infinitesimals as "fiction" is employed precisely to distinguish his syncategorematical infinitesimal from other mathematical terms. The logicist framework employed by Ishiguro simply cannot accommodate these complexities and lands her in an empty judgment of Leibniz's infinitesimals. In short, if all mathematical terms are made through contextual reference, the use of contextual reference neither adds anything to our understanding of Leibniz's admission of infinitesimal methods nor allows us to understand the difference marked by the "fictional" nature of these terms. I argued that a closer look at the original framework of Leibniz's infinitesimal method in the problem of quadrature reveals that Leibniz faced a problem of rigor and justification that was at a certain distance from this debate between actual and fictional infinitesimal. In any case, Leibniz already clearly rejected anything like an actual infinitesimal entity. The complexity of his fictional infinitesimal is only fully appreciated through its role in the improvement that he brought to an Archimedean style argument for the problem of quadrature. The drive to focus on the reducibility of infinitesimal fictions in this demonstration in the *Quadratura* occludes what Leibniz saw as the problem in this same demonstration. In turn, we see that the problem that concerned Leibniz from the perspective of a "toolbox" conception of the foundation of these infinitesimal fictions brings much more complexity with regard to the role that such fictions play. This purely reductive view of infinitesimals overshadows the constraints that Leibniz saw himself as facing and overcoming in his quadrature method.

References

¹ Leibniz, G.W., Letter to Pinsson, 29 August 1701, in *Sämtliche Schriften und Briefe*, eds. Berlin-Brandenburgische Akademie der Wissenschaften and Akademie der Wissenschaften zu Göttingen (Berlin: Akademie Verlag, 1923–), Ist series, vol. xx, 494.

² The definition of an Archimedean expression in quantity or magnitude, an important term that for the remainder of this text, will be defined in its Euclidian manner as quantities that "have a ratio to one another which are capable, when multiplied, of exceeding one another." Please note that although I am citing Euclid's formulation of the Archimedean property, it should not be confused with Euclid's law which asserts a relation between part and whole." See Euclid, *Elements*, trans. T.L. Heath, ed. D. Densmore (Santa Fe: Green Lion Press, 2002), 99.

³ Leibniz demonstrates a commitment to the infinite character of nature or the world quite early on. This applies equally to treatments of the extended world, the predicate-parts of substances as well as to the later treatment of the aggregation of monads in extended space qua composite body. In its various versions from Leibniz's youth to his maturity, the infinity of nature or the world may be understood as "concrete" or even "actual" insofar as it describes actual things. This "actual" or "concrete" division of the world into infinite parts does not of course imply that Leibniz held, in his maturity, an actual infinite sum, number or totality. When Leibniz speaks of aggregates as concretely divided to infinity in, say, the *Monadology*, I understand this as consistent with a syncategorematic division. However there is, without no

doubt, a tension that remains between the notion of the infinite analysis of an extended aggregate thing into partless and extensionless monads and the explicit rejection of an actual infinite. My assertion of a dissymmetry here points to tensions such as these that arise frequently in Leibniz's metaphysical works. Leibniz's use of the infinite in metaphysics (especially when describing the division of extended nature) may be consistent with the syncategorematic infinite but comes into conflict with a reduction of the syncategorematic to the finite. A good example of this is Leibniz's 1678 "Actu infinitae sunt creaturae" where he explicitly states the infinite nature of bodies through division. Cf. Leibniz. A VI, 4 N. 266.

⁴ Katz M., and Sherry, D., "Leibniz's Infinitesimals: Their Fictionality, Their Modern Implementations, and Their Foes from Berkeley to Russell and Beyond", *Erkenntnis*, (online first on 23 April 2012): 1-55, 51. Cf. Bos, H.J.M. "Differentials, Higher-order Differentials and the Derivative in the Leibnizian Calculus," *Archive for the History of the Exact Sciences* 14 (1974): 65-66.

⁵ Leibniz, G.W., *Leibnizens mathematische Schriften*, ed. C.I. Gerhardt (Berlin: Asher; Halle: Schmidt, 1848-1863), 7 vols., vol. V, 322, as quoted in Bos, H.J.M.(1974), 64.

⁶ Katz, M. and Sherry, D.(2012), 13.

⁷ Leibniz, G.W., *Quadrature arithmétique ducercle, de l'ellipse et de l'hyperbole et la trigonométrie sans tables qui en est le corollaire*, ed. and trans. M. Parmentier, ed. Latin text E. Knobloch (Paris: Vrin, 2004), hereafter *Quadratura*.

⁸ Knobloch, E., "Leibniz's Rigorous Foundations of Infinitesimal Geometry by Means of Riemannian Sums", *Synthese* 133: (2002): 59-73, 59.

⁹ Leibniz, G.W., *Mathematische Schriften*, vol. IV, 110.

¹⁰ I rely on Paolo Mancosu's account of this debate. See Mancosu, P., *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (Oxford: Oxford University Press, 1996), 165-177.

¹¹ Leibniz, G.W., *Philosophical Papers and Letters*, ed. and trans. L. Loemker, 2nd edition (Dordrecht: Kluwer Academic Publishers, 1989), 544.

¹² Leibniz, G.W., *Mathematische Schriften*, vol. IV, 91.

¹³ Leibniz, G.W., *Sämtliche Schriften und Briefe*, VIth series, vol. iii, n°11; Leibniz, G.W., *The Labyrinth of the Continuum*, ed. and trans. R.T.W. Arthur (New Haven and London: Yale University Press, 2001), 9.

¹⁴ Leibniz, G.W., *Nouveaux essais sur l'entendement humain*, (*Sämtliche Schriften und Briefe*, VIth series, vol. vi, 157), cited from Leibniz, G.W., *New Essays on Human Understanding*, ed. and trans. P. Remnant and J. Bennett. (Cambridge: Cambridge University Press, 1996), 124.

¹⁵ Arthur, R.T.W., "A Complete Denial of the Continuous? Leibniz's Law of Continuity", [Online] Available via: <http://www.humanities.mcmaster.ca/~rthur/papers/LeibCant.pdf>, 7.

¹⁶ A treatment of the relation between the three senses of the infinite is undertaken by O. Bradley Bassler. While I do not follow him in adopting a terminology of the "indefinite" or "parafinite," I do share his general framework of reading Leibniz's introduction of the syncategorematic infinite in the *New Essays* and elsewhere. In this discussion, I merely aim to show that the distinction between the three senses of the infinite exists, and that the syncategorematic infinite should not be reduced to either of the two other senses. See Bassler, O.B., "Leibniz on the indefinite as infinite", *The Review of Metaphysics* 51 (June 1998): 849-874, here 856-857.

¹⁷ My charge is that Ishiguro's reduction of the status of infinitesimals to mathematical objects as such occludes the peculiar status that infinitesimals hold. Based on her argument, one could ultimately assert *either* a finitist reduction *or* some alternative non-finitist position. The problem

of equivocation in Ishiguro that I aim to point out is that the treatment of Leibniz's infinitesimal through her logicist framework and her direct invocation of a meta-mathematical treatment of infinitesimal on par with any mathematical term whatsoever provokes equivocation. The position advanced by Sherry and Katz argues that this approach tends toward a finitist reduction. Although I judge that Ishiguro's logicist interpretation of the syncategorematic does not strictly entail such a finitist reduction, my point is that her framework results in the kind of equivocation that results in the sorts of interpretations that Sherry and Katz aim to refute.

¹⁸ With respect to mathematical objects, a meta-mathematical claim is not simply an ontological one. In the case of the logicist position asserted by Ishiguro however the treatment of the object-nature of mathematical terms is precisely its meta-mathematical status. This is the legacy of the *Grundlagenkrise der Mathematik* in Ishiguro's interpretation of Leibniz that I will address in the below.

¹⁹ Robinson, A., *Non-Standard Analysis* (Amsterdam: North-Holland Publishing Company, 1970).

²⁰ Katz, M and Sherry, D. (2012) dedicate their article to demonstrating the continuity between Leibniz and Robinson that has been covered over by an overly hasty reduction of infinitesimals in Leibniz to finite quantities. They remark that, "We shall argue that Leibniz's system for the calculus was free of contradiction, and incorporated versatile heuristic principles such as the law of continuity and the transcendental law of homogeneity which were, in the fullness of time, amenable to mathematical implementation as general principles governing the manipulation of infinitesimal and infinitely large quantities. And we shall be particularly concerned to undermine the view that Berkeley's objections to the infinitesimal calculus were so decisive that an entirely different approach to infinitesimals was required." (Katz, M. and Sherry, D. (2012), 4.).

²¹ The definition and use of limits by Cauchy got rid of a number of ambiguities surrounding the infinitesimal by relating it to the idea of a function that indefinitely approaches zero. Employing this idea, Cauchy removes the need to appeal to geometric intuitions where confusions about the infinitely small magnitude often invited confusing appeals to an "infinitesimal" entity. See Boyer, C., *The History of the Calculus and its Conceptual Development* (New York: Dover Publications, 1959), 272-274.

²² In his well-known attack on the notion of infinitesimals, George Berkeley balked at the idea of the infinitesimal quantity or increment, an attack launched principally against Newton. In a comment directed more against Newton than Leibniz, he characterizes these terms as "the ghosts of departed quantities." As Abraham Robinson and other contemporary mathematicians in the domain of infinitesimals remark, Berkeley's criticism was aimed at a theory of infinitesimals that implied, directly or indirectly, the idea of an infinitesimal number or entity. As Robinson notes, Leibniz was often careful to avoid this implication, but he did not lay any clear basis to sort out the issue. See Robinson, A. (1970), 265-266.

²³ Robinson, A. (1970), 282.

²⁴ Robinson, A. (1970), 282; See Leibniz, G.W., *Sämtliche Schriften und Briefe*, Ist series, vol. xx, 494.

²⁵ Ishiguro, H., *Leibniz's Philosophy of Logic and Language*, 2nd edition (Cambridge: Cambridge University Press, 1990), 83.

²⁶ Ishiguro, H. (1990), 86.; Leibniz, G.W., *Mathematische Schriften*, vol. IV, 91; Leibniz, G.W., *Philosophical Papers and Letters*, 542-554.

²⁷ This controversial position advanced by Ishiguro helps us grasp what is precisely at stake for a logicist reading of Leibniz's infinitesimals: de-emphasizing of intra-mathematical distinctions for the sake of meta-mathematical homogeneity.

²⁸ Ishiguro argues, "One cannot however, like Boyer, criticize Leibniz for being 'unable to explain the transition from finite to infinitesimal magnitudes' in his consideration of the ratio between dx and dy . As we saw, there are no actual infinitesimal magnitudes. There are only finite magnitudes which we can take as small as we wish. The quotient dy/dx is defined, nevertheless, for all the values of dx , fictiously including the limit." Ishiguro, H. (1990), 95.

²⁹ Please keep in mind that this interpretation of "incomparability" is Ishiguro's. She argues, "Often when Leibniz uses the word 'incomparable', he is ... thinking in the line of the traditional notion of incomparable non-homogenous quantities [...] For example, the length of a line and an area of a plane could not, it was thought, be compared with each other since entities of different dimensions were said not to have homogeneous quantities." In the same argument Ishiguro continues to qualify that Leibniz's notion of what is homogeneous does not directly depend on geometrical intuitions of dimensionality but on a logical distinction between things that are comparable and things that are not. As such, Ishiguro argues that nothing can be "incomparably" larger or smaller since larger or smaller already assume some notion of comparability. Ishiguro then concludes, "[A]dding a line to a surface does not increase the surface, but this is surely not because the line is incomparably smaller than any surface." Ishiguro, H. (1990), 88.

³⁰ Ishiguro, H. (1990), 90; Leibniz, G.W., *Philosophical Papers and Letters*, 546.

³¹ Ishiguro, H. (1990), 90; Leibniz, G.W., *Philosophical Papers and Letters*, 543.

³² Ishiguro, H. (1990), 90; Leibniz, G.W., *Philosophical Papers and Letters*, 92.

³³ Ishiguro, H. (1990), 90; Leibniz, G.W., *Philosophical Papers and Letters*, 92.

³⁴ Bos's landmark work in evaluating Leibniz's calculus is one of the many sources arguing that Leibniz never held a limit concept of the infinite and infinitesimal. While I do not employ Bos's comparison of Leibniz's calculus and modern calculus undertaken in "Differentials, higher-order differentials and the derivative in the Leibnizian Calculus," he has argued here and elsewhere that Leibniz's calculus stands on its own and does not require later mathematical developments to either vindicate or reappraise it. In this same article, he attempts to provide two different ways of reading "foundation" in Leibniz's methods for the calculus. He distinguishes between the use of the differential algorithm through the infinite series and the use of infinitesimals as entities. While I do hold these two sorts of justifications or "foundations" apart, Bos's fundamental argument in this text argues that Leibniz's development of the *method* of the calculus is rooted in the "extrapolation" of the infinite and infinitesimal from the finite case. While Bos does note that Leibniz's legacy left the status of the terms of his calculus open to misinterpretation in its lack of a foundation, he also notes that Leibniz was careful not to treat his infinite and infinitesimal terms as independent definite entities. He warns, "The common concern of historians with the difficulties connected with the infinite smallness of differentials has distracted attention from the fact that in the practice of the Leibnizian calculus, differentials as single entities hardly ever occur. The differentials are ranged in sequences along the axes, the curve and the domains of the other variables; they are variables, themselves depending on the other variables involved in the problem, and this dependence is studied in terms of differential equations." (Bos, H.J.M. (1974), 17). See also Bos, H.J.M., "The Fundamental Concepts of the Leibnizian Calculus," in *Lectures in the History of Mathematics* (Providence, RI: American Mathematical Society, 1993), 83-99.

³⁵ Ishiguro, H. (1990), 96.

³⁶ Ishiguro, H. (1990), 98.

³⁷ Ishiguro, H. (1990), 100.

³⁸ My use of Ishiguro as a representative of a “reductive” reading of Leibniz’s syncategorematic infinite may appear to create a strawman argument insofar as it does not reflect more current Leibniz research. This is partly true but my aim here is to criticize the logicist means by which Leibniz’s infinitesimals are approached rather than to mistake Ishiguro’s particular argument here for current Leibniz research. A number of articles by H. Breger, R. Arthur, S. Levey, D. Garber and D. Rutherford which have informed the research of this article have employed the notion of the syncategorematic infinite and infinitesimal in a non-reductive way that supports or is at least consistent with my approach. For the sake of brevity, these articles are not discussed here. Yet in Ishiguro we find not only a highly influential but also a generalized approach to Leibniz’s philosophy through the notion that logic constitutes a foundation for metaphysics, epistemology and mathematics. The general aim to demonstrate the flaws of a logicist reading of Leibniz which feeds into a number of different interpretations is hence not a strawman. Rather, an explicit denial of the logicist means of framing the question of the role of Leibniz’s (syncategorematic) infinite and infinitesimal is needed to explicitly create a context where these terms are balanced between metaphysical, epistemological and mathematical considerations.

³⁹ Benacerraf, P., “What numbers could not be”, *The Philosophical Review*, 74,1 (Jan 1965): 47-73.

⁴⁰ Marquis, J.P., “Category Theory and the Foundations of Mathematics: Philosophical Excavations”, *Synthese* 103, 3 (1995): 421-447.

⁴¹ The first 7 propositions of the *Quadratura* lay out a first methodological layer that Leibniz uses as a point of departure to other problems in his mathematical treatise of 51 propositions. Much later in the treatise, Leibniz uses this first layer of constructions and arguments to develop (in proposition 48) a formulation of the famous alternating infinite series (for $\pi/4$). My reading of the *Quadratura* is largely informed by Eberhard Knobloch, Marc Parmentier and David Rabouin. Cf. Knobloch, E., “Generality and Infinitely Small Quantities in Leibniz’s Mathematics –The Case of his Arithmetical Quadrature of Conic Sections and Related Curves”, in *Infinitesimal Differences*, ed. U. Goldenbaum and D. Jesseph, (Berlin/New York: Walter de Gruyter, 2008), 171-183; Parmentier, M., “Démonstrations et infiniment petits dans la *Quadratura arithmetica* de Leibniz”, *Revue d’histoire des sciences*, 54, 3 (2001): 275-289; Rabouin, D., “Infini mathématique et infini métaphysique: D’un bon usage de Leibniz pour lire Cues (... et d’autres)”, *Revue de métaphysique et de morale*, 70 (2011): 203-220.

⁴² Leibniz, G.W., *Quadratura*, 34.

⁴³ Figure taken from Knobloch E. (2008), 182.

⁴⁴ Figure modified from Knobloch, E. (2008), 182.

⁴⁵ Leibniz, G.W., *Quadratura*, 69 [my translation].

⁴⁶ Knobloch, E. (2002), 63.

⁴⁷ See Rabouin, D. (2011), 214.

⁴⁸ Leibniz, G.W., *Quadratura*, 69-71 [my translation].

⁴⁹ Levey, S., “Archimedes, Infinitesimals and the Law of Continuity: On Leibniz’s Fictionalism”, in Goldenbaum, U. and Jesseph, D. (2008), 107-133, 117.

⁵⁰ Levey, S. (2008), 119.

⁵¹ See Mercer, C., *Leibniz’s Metaphysics: its Origins and Development* (Cambridge: Cambridge University Press), 2002).

⁵² See Mancosu, P. (1996), 35.

⁵³ Mancosu, P. (1996), 35.

⁵⁴ See Mancosu, P. (1996), 10-15.

⁵⁵ See Mancosu, P. (1996), 24-28.

⁵⁶ Mancosu, P. (1996), 35.

⁵⁷ Mancosu, P. (1996), 128.

⁵⁸ Levey, S. (2008), 118.