FROM CAUSES TO RELATIONS: THE EMERGENCE OF A NON-ARISTOTELIAN CONCEPT OF GEOMETRICAL PROOF OUT OF THE *QUAESTIO DE CERTITUDINE MATHEMATICARUM*

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Abstract. Insofar as many Renaissance thinkers regard Aristotelian philosophy of science as the framework for their understanding of mathematics and its proofs, they consider geometrical proofs as syllogisms using causes. Furthermore, they identify geometrical proofs as *demonstrationes potissimae*, which are a kind syllogism that provides both the cause and the effect of an event. By questioning this assumption, Piccolomini initiates the so-called *Quaestio de certitudine mathematicarum*. Several scholars agreed with him. Others either maintained that mathematical proofs are *demonstrationes potissimae* or tried to prove that at least some mathematical proofs satisfy the conditions for being *demonstrationes potissimae*. Despite their differences in detail, all participants in the debate recognized Aristotelian scientific theory as the norm. Yet even traditionally Aristotelian answers take on a new meaning by virtue of a new context. This marks the birth of a genuinely new debate which has unwittingly left its Aristotelian roots behind. As a result, geometrical proofs are no longer thought of as being based on causes or principles of being, but on the relationship between the different figures. Such a relationalism opens up the possibility of further development of mathematics.

Keywords: mathematics, proof, geometry, Aristotelianism, Renaissance, certainty

Introduction: Relational and Causal Understanding of Mathematics

In his *Philosophical Essays concerning Human Understanding* (1748), David Hume distinguishes two kinds of objects of human reason, namely relations of ideas and matters of fact. The propositions in geometry, algebra and arithmetic are of the first kind. Unlike the matters of fact, the relations of ideas are demonstratively certain.

> *That the Square of the Hypothenuse is equal to the Squares of the two Sides*, is a Proposition, that expresses a Relation betwixt these Figures. *That three*

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> *times five is equal to the half of thirty*, expresses a Relation betwixt these Numbers. Propositions of this Kind are discoverable by the mere Operation of Thought, without Dependance on what is any where existent in the Universe. Tho' there never were a true Circle or Triangle in Nature, the Propositions, demonstrated by Euclid, would for ever retain all their Truth and Evidence.[1]

What Hume expresses here, I want to call a *relational understanding of mathematics*, which is characterised by two main features. First of all, mathematical propositions are based on the internal relations between the mathematical objects. Secondly, this implies a flexible stance towards mathematics' ontological foundation. A historically important counter-model is *causal understanding of mathematics*. Its point of reference is Aristotle's theory of science as set out in the *Posterior Analytics*: "According to Aristotle, full-fledged scientific knowledge of something requires understanding its necessitating causes; this knowledge is produced or best manifested by demonstrative syllogism."[2]

Knowledge here means knowledge of the cause in the sense of Aristotle's four causes (*aitia*). The four causes can be regarded as four types of *explanations why* the thing in question is how it is.[3] Unlike the relationalism, the Aristotelian understanding of mathematics implies a strict ontological foundation for mathematics and mathematical proofs, insofar as mathematical propositions are based on such causes. More precisely, the Aristotelians regard the mathematical objects themselves (and not the relations between them) as such causes. Via abstraction, mathematical objects are dependent on their instances in the world. The relationalists base mathematical proofs on the relations between the figures and on the particular construction of each figure; the Aristotelians base mathematical proofs on the mathematical objects gained via abstraction.

From the 13th century to the Renaissance, Aristotelian philosophy of science is the umbrella concept for the understanding of mathematical method and proof, providing its terminological framework (although scholars in the period combine the Aristotelian concept of science with other approaches).[4] In the 17th century, the Aristotelian understanding of mathematics gets replaced by a relational understanding. The latter is a prerequisite for various scientific achievements of modern times.

One such scientific achievement is the advent of non-Euclidean geometry. Because Aristotelian mathematics relies on abstraction from experience (and experience shows non-Euclidean geometry to be (psychologically) impossible[5]), it cannot allow for the possibility of a non-Euclidean geometry. However, with reference to its logical consistency, non-Euclidean geometry is logically possible.[6] Thus, in the long run, the relational understanding of mathematics was a preliminary condition for the recognition of the non-Euclidean geometry. Based on the relational understanding of geometry, we can, therefore, accept the logical possibility of non-Euclidean geometry.[7] In the shorter term, the relational understanding of mathematics enabled some scientific achievements of the 17th century, particularly the mathematization of non-mathematical sciences and the algebraisation of geometry. Within strict Aristotelianism, all scientific disciplines, however, have their unique subject areas, and hence the methods of one discipline cannot be applied to the subject area of another discipline. Therefore, it was the break with Aristotelian understanding of science and mathematics that allowed for several breakthroughs, not least the so-called "scientific revolution."[8]

My paper deals specifically with the break away from the Aristotelian causal theory of geometry and geometrical proof. I want to demonstrate that this break emerged within the scholastic Aristotelianism itself in the second half of the 16th century – in the prehistory of scientific revolution.

During the Renaissance there was a growing interest in mathematical method. On the one hand, this was caused by the mathematical problems that craftsmen had to deal with as a consequence of their practical needs. On the other hand, it was a product of renewed interest in non-Aristotelian ancient mathematical writings (such as Euclid's Elements and Proclus' commentary on Euclid), which were extensively published in new, translated editions at the time.[9] Many actors were important in such developments, but the school of Padua played a decisive role within the discussions about method and mathematics.[10] For the Paduan scholars, as for most Renaissance thinkers, Aristotelian philosophy of science (rather its scholastic version than the original one) is the umbrella concept for their understanding of mathematical method and proof. In the early modern period, many philosophers of mathematics either regarded geometrical proofs as syllogisms or thought that they should be reformulated as syllogisms.[11] In most cases, geometrical proofs were equated with a specific kind of Aristotelian proof, namely the *demonstratio potissima* (a kind of syllogism that provides both the cause and the effect of an event; more detailed explanation in chap. 2.2 below). As a critical response to this approach, the Paduan philosopher Alessandro Piccolomini (1508–1579) initiated a debate that came to be called the *Quaestio de certitudine mathematicarum*. This is the starting point for my investigation of the shift from causes to relations in mathematical thinking.

The *Quaestio de certitudine* deals with three interrelated questions. First, it questions the certainty of mathematics in general. Since mathematical certitude is traditionally justified by the special character of mathematical proofs, the initial argument focuses on the second question: whether or not there is a place in mathematics for the *demonstratio potissima*. (I use the epithet "initial" in order to distinguish the 16th century *quaestio* from its revival in the 17th century.) In the course of the debate, these initial questions increasingly lose their relevance. More and more, attention shifts to the third question: "whether the actual procedure of geometers in proving theorems and solving problems could be reconciled with Aristotle's description of a demonstrated science."[12] Such a procedure was mainly based on Euclid's *Elements*. Therefore, the real subject of the discussion is the incompatibility of Euclidean geometry with the Aristotelian understanding of science.

The initial debate is a symptom of the contradictions within the traditional preconditions of the understanding of geometry, namely the lack of distinction between Aristotelian proofs and geometrical method.[13] In fact, the debate about the certainty of mathematics shows the inadequacy of using the criteria of Aristotelian proof theory to describe mathematical proofs. From a present-day perspective, the debate is based on completely misguided assumptions. No consensus was reached, and conflicts were not resolved. However, although all participants in the debate remained firmly within the Aristotelian framework, a new concept of geometrical proof *ex negativo* emerged in the discussion. The initial discussion has a scholastic nature, admittedly, but as it progresses, the Aristotelian understanding of science grows increasingly vague. This lay the groundwork for a non-Aristotelian understanding of geometry with attempts to reformulate Euclidean geometry by means of Aristotelian tools. As understood by an Aristotelian, geometry is supposed to explain single geometric figures in terms of their unique causes. The new concept of geometrical proof, however, emphasizes the relations between figures and focuses the debate on the practice of geometrical constructions. This concept of geometrical proof carries changes in the understanding of geometry in general within itself. When scholars such as Gassendi, Wallis, Hobbes and Barrow take up the debate again in the 17th century, they are no longer interested in the question of whether there are *demonstrationes potissimae* in geometry, and they are equally uninterested in rescuing the Aristotelian framework. Insofar as they adopt some questions and arguments of the initial debate, they take on the relational understanding of geometry.

I am going to trace the shift from causes to relations in three steps. In the first, I will outline the Aristotelian background of the debate, in particular, the traditional justifications of mathematical certainty. More specifically, I shall set out the features of the *demonstratio potissima*. In the second step, I am going to sketch the main positions in the *Quaestio de certitudine mathematicarum*. In the third, I want to display the new concept of geometrical proof which looms up in the discussion. In the course of this, I will sketch out the *quaestio*'s revival in the 17th century and contextualise it: I will consider the relational understanding of geometrical proof in the context of the overall development in mathematical thinking of this period.

2. The Background: Aristotelian Justifications of the Certainty of Mathematics

In the philosophy of mathematics during the early modern period, we can identify two justifications for mathematical certainty.[14] The first strategy justifies the *objective certainty* of mathematics by means of the ontological status of their entities (2.1). The second strategy deduces the *subjective certainty* of mathematics from the character of mathematical proofs (2.2). The second strategy is the more important for the purposes of this paper. Hence, after giving a short explanation of the first, I will concentrate on the second strategy.

2.1. The Ontological Status of Mathematical Entities

The first strategy is used by, for example, Thomas Aquinas.[15] Following Aristotle, he regards mathematical entities as *abstractions* based on sense experience. From the ontological status of mathematical entities, Thomas concludes that mathematics is more certain than both natural philosophy and theology. This is so, because, unlike natural philosophy, mathematics does not deal with matter and motion; unlike theology, mathematics considers entities which are given to the senses and to the imagination. To put it in a nutshell, insofar as mathematical entities are created by abstraction, they have the highest level of clarity and evidence.

2.2. The Use of the *Demonstratio Potissima*

The second strategy bases mathematical certainty on the characteristics of the proofs used in mathematics. In the *Posterior Analytics*, Aristotle envisions science as true knowledge gained via reasons or causes.[16] Furthermore, he maintains that mathematical disciplines produce proof by use of a syllogism *in the first figure*,[17] which has the following form:

maior: middle term (M) – predicate (P).
minor: subject (S) – middle term (M).
conclusio: subject (S) – predicate (P).

In the early modern period, only a few philosophers of mathematics drew a distinction between geometrical proof and the Aristotelian syllogism.[18] Most of them classify geometrical proofs as *demonstrationes potissimae*, which are regarded as the highest and most certain type of proof. I will explain this type of proof by comparing it with the two other types, namely the *demonstratio quia* and the *demonstratio propter quid*.[19]

All three types of proof are regarded as a syllogism in the first figure. The *demonstratio quia* infers the cause from its effect. This kind of proof can be illustrated by an example from the *Posterior Analytics*.[20]

maior: Non-twinkling heavenly bodies (M) are near earth (P).
minor: Planets (S) are non-twinkling heavenly bodies (M).
conclusio: Planets (S) are near earth (P).

Its middle term is the unique effect (*effectus proprius*). It signifies the (observed) effect, namely that these heavenly bodies do not twinkle. By rearranging this syllogism, we get the *demonstratio propter quid*.

maior: Heavenly bodies which are near earth (M) do not twinkle (P).
minor: Planets (S) are heavenly bodies which are near earth (M).
conclusio: Planets (S) do not twinkle (P).

The *demonstratio propter quid* infers the effect from its proximate cause.[21] Its middle term signifies the proximate cause of the effect, in our example the proximity to the earth. (Of course, the middle term of such a proof is understood as one of the four causes in the Aristotelian sense.)

At this point, Aristotle leaves us. He distinguishes only these two kinds of proof. But following Averroes, Aristotelians assume a third type of proof, namely *demonstratio potissima*.[22] Such a proof infers the effect (*esse*) and the cause (*the propter quid effectus*) from fundamental premises.[23] It is a syllogism that provides both the cause and the effect of an event by using a middle term which specifies the proximate cause of the effect in a unique way.[24]

The primary, as well as the secondary literature, usually content themselves with giving an abstraction explication of the *demonstratio potissima*, but they do not provide an actual example of a *demonstratio potissima*.[25] There are two possible interpretations. Following the first interpretation, we get the *demonstratio potissima* by another rearrangement of the syllogisms above. According to the second interpretation, the *demonstratio potissima* is just a variant of the *demonstratio propter quid*. There is only one possibility for further rearranging the syllogisms above:

maior: Planets (M) are near the earth (P).
minor: Not twinkling heavenly bodies (S) are planets (M).
conclusio: Not twinkling heavenly bodies (S) are near the earth (P).

Indeed, the resulting syllogism does not fulfil all requirements for being a *demonstratio potissima*. Therefore, it would be more reasonable to assume that the *demonstratio potissima* is nothing but a variant of the *demonstratio propter quid* which is only accidentally distinguished from the *demonstratio propter quid* in the proper sense.[26] If so, it would be a *demonstratio propter quid* in which the effect is unknown. Being a *demonstratio potissima* would depend on the previous knowledge of the recipient.

Despite the problems concerning the interpretation of this kind of proof, Aristotelian philosophers of mathematics base the certainty of mathematics on its use of the *demonstratio potissima* as the most certain type of proof. They regard it as the most certain type of proof because it provides us with the cause and its effect at once. However, the equivalence of mathematical proof and *demonstratio potissima* was essentially contested.[27] It was this that led Piccolomini to initiate the debate, *Quaestio de certitudine mathematicarum*. Within the framework of this debate, even the traditionally Aristotelian answers take on a new meaning by virtue of a new context. This marks the birth of a genuinely new debate which has unwittingly left its Aristotelian roots behind. I am not interested in the result of the initial Aristotelian debate, not least because there was no real final solution. Rather, my interest lies in the relational understanding of geometry and geometrical proof which looms in the background of the debate.

3. Main Positions in the Initial Debate About Certainty in Mathematics

The subject of the *Quaestio* is the question of the certainty of mathematics. Nevertheless, the initial debate focuses on the question of whether geometrical proofs can be identified as *demonstrationes potissimae*. From a logical point of view, three possible positions can be distinguished in the debate.[28] The first group defends the identification thesis: according to the traditional position (e.g., Hieronymus Balduinus, Jacob Schegk (1511–1587)), *all* geometrical proofs are *demonstrationes potissimae*. The second group denies the identification of geometrical proofs and *demonstrationes potissimae* (2.1). The critics (e.g., Alessandro Piccolomini (1508–1579), Simon Simonius (1522–1602), Benedictus Pererius (1535–1610), the Jesuits of Coimbra, Martin

Smiglecius (1562/1564–1618)) claim that *no* geometrical proofs are *demonstrationes potissimae*. The moderate defenders (e.g., Franciscus Barocius (1537–1604), Joseph Blancanus (1566–1624)), as the third group, maintain that at least *some* geometrical proofs are *demonstrationes potissimae* (2.2). After Piccolomini's *commentarium de certitudine mathematicarum disciplinarum* (1547), the first position maintained very few proponents.[29] The debate mainly takes place between the critics and the moderate defenders. To characterize these positions in more detail, I will focus less on their specific arguments and more on their assumptions and the implications of their arguments. I will start with the positions of the critics.

3.1. Critics: No Mathematical Proofs are *Demonstrationes potissimae*

As one of the critics, Piccolomini's arguments define the debate.[30] The critics of the identification thesis often further refine his arguments and investigate their implications. Piccolomini shows that geometrical demonstrations are no proofs by any of the four causes.[31]

a) Geometrical demonstrations are not proofs by efficient cause (*causa efficiens*) because mathematics does not deal with action.[32] Many of Piccolomini's arguments are based on the assumption that the geometrical objects understood as pure quantity (*quantitas*) have no relation to action (*actio*).[33] Simonius (*Antischegkianorum Libernus*, 304 and 310) follows this argument in a very peculiar way. He believes that there are *demonstrationes potissimae* used in mathematics with the formal cause as middle term. But he regards only the proofs from the efficient cause as the most perfect.[34]

b) Geometrical demonstrations are not proofs by final cause (*causa finalis*).[35] Of course, mathematics does have purposes, insofar as it is useful for various applications. But there are no final causes *within* mathematics. Piccolomini argues that only activities have purposes and therefore final causes. But mathematical objects are immutable. Where there is no change, there can be no purpose (of change).

c) Geometrical demonstrations are not proofs by *material cause* (*causa materialis*) because there is no *real* matter (*materia realis*) in mathematics.[36] Mathematics just deals with *intelligible* matter (*materia intelligiblis*) created by abstraction.

d) Geometrical demonstrations are not proofs by formal cause (*causa formalis*). Since Piccolomini attacks the traditional view here, its refutation takes up the largest room within his arguments by far. Looking at the progress of the debate, we can identify two influential arguments against the use of formal causes in geometrical proofs.

To begin with, the middle term of every *demonstratio potissima* has to be the definition of the subject or of its property (*definitio vel subiecti vel passionis*).[37] Piccolomini shows that geometrical proofs do not use such a middle term, by referring to Euclid's demonstration that the angle sum in a triangle equals two right angles (see below fig. 1).[38] The demonstration (Euclid I.32) in short is as follows. AB is parallel to CE. Therefore, the alternate angles BAC and ACE equal one another and the corresponding angles ABC and ECD equal one another. Accordingly, the angle ACD equals the sum of the angles BAC and ABC; and thus the sum of the interior angles equals the sum of the angles ACD and ACB. Since the sum of the angles ACD and

ACB equals two right angles, the sum of the interior angles equals the sum of two right angles.

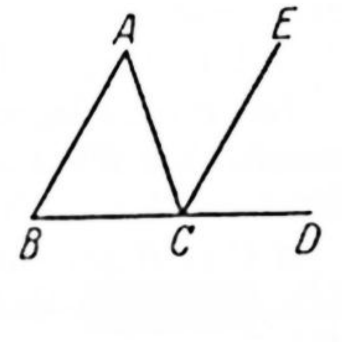

Fig. 1

Obviously, this demonstration makes use of the exterior angle. Proclus already questions whether this proof uses (real) causes.[39] Following Proclus, Piccolomini shows that the exterior angle is neither a definition of the triangle itself nor of one of its properties. The exterior angle is not part of the triangle's definition. Even if the exterior angle did not exist, it would still be a triangle.

Furthermore, the *demonstratio potissima* requires a middle term which signifies a unique and proximate cause.[40] Piccolomini emphasizes that there is *no hierarchy of priorities* between the mathematical properties with respect to their dependencies. Instead, one theorem can be proven by different premises.[41] One middle term used in a proof is neither more unique nor more proximate than another possible middle term.

While Piccolomini does indeed justify the certainty of mathematics by means of the nature of its entities,[42] the critical part of his argumentation was significantly more influential. Several scholars agreed with Piccolomini in one respect or another.[43] Alongside various refinements of Piccolomini's observations, *two main arguments* evolved. Each is deeply connected with the other.

The first argument is based on the distinction between the *principle of Being* (*principium essendi*) and the *epistemological principle* (*principium cognoscendi*).[44] Geometric proofs do not use principles of Being insofar as they do not use real causes. Instead, they use only the second one, the epistemological principles, in the sense that the proofs rely on reasons for understanding. We understand or comprehend a figure's properties by its construction. Indeed, a strict Aristotelian does not regard its construction as the cause of its properties. The construction only provides us with a principle of understanding. By contrast, a principle of Being of a figure's properties would be its cause in an ontological sense.

The second argument is based on the distinction between the *essence* of a geometric figure and its *relations* to other figures. Many of Euclid's proofs demonstrate properties of one geometric figure by using its relations to other figures. But a figure's relation to other figures does not belong to its essence; nor are any of these relations unique and proximate causes, since there is no hierarchy of priorities between mathematical properties with respect to their dependencies. Thus, geometric proofs do not follow from the essences of figures. I am going to illustrate this argument in reference to Euclid's construction of an equilateral triangle (see below fig. 2).[45] The task is to construct an equilateral triangle on a given finite straight line AB. To do this, we have to describe the circle BCD with centre A and radius AB and the circle ACE with centre B and radius BA. Their point of intersection C has the same distance AB to A and to B. Therefore, the triangle ABC is equilateral.

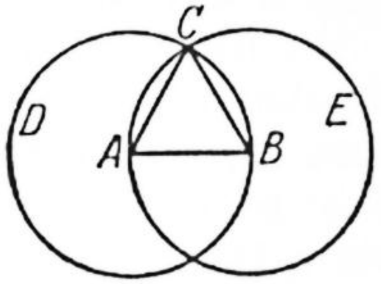

Fig. 2

Euclid uses here the circle, or its definition, in order to construct the equilateral triangle and to demonstrate its properties. In this respect, being equilateral is not proven by the essence of the triangle but by its relation to other figures. Smiglecius (1562/1564–1618) refines this argument. In my English translation, I make its syllogistic structure explicit. My primary interest is not the *conclusio* but the *minor*.

> [*maior*:] In the *demonstratio potissima*, the cause of the property or characteristic is the essence of the subject from which the properties originate.
> [*minor*:] In Mathematics, the properties are not derived from the essence of the subject, but from the relations to other figures.
> [*conclusio*:] Geometrical proofs do not demonstrate properties by using the real cause of the essence or Being.[46]

Pererius, among others, radicalized Piccolomini's theses and arguments by denying mathematics the status of science.[47] Geometrical proofs do not prove by real causes (in the sense of principles of being), and in Aristotelian philosophy of science, proving by causes is a requisite for being a science. Geometrical proofs do not fulfil this condition and therefore mathematics is not a science in an Aristotelian sense. This is the point at which the moderate defenders enter the debate.

3.2. Moderate Defenders: Some Mathematical Proofs are *Demonstrationes potissimae*

Many philosophers in the 16th and early 17th centuries did not want to accept the consequence that mathematics (the prime example of a science) should not be a science at all. So, the more moderate defenders, such as Barozzi (aka Barocius) and Blancanus (aka Biancani) tried to prove that at least *some* mathematical proofs satisfy the conditions for being a *demonstratio potissima*.[48] In order to do this, they have to show that the middle terms of these proofs signify the unique and proximate cause of the property in question. Commonly, they regard the middle terms as definitions which denote formal or material causes.[49]

Blancanus discusses the two paradigms of geometrical proofs in the debate and comes to the following result. In his opinion, the equilateral triangle (Euclid I.1) is proven by formal cause, insofar as he regards the definition of the circle as a formal cause.[50] He takes the proof of the triangle's angle sum (Euclid I.32) as a proof by material cause, insofar as it is a conclusion from the parts to the whole.[51]

His arguments are based on a very peculiar notion of definitions in geometry. While the critics usually conceive of such definitions as being *nominal*, Blancanus argues that in geometry definitions are nominal as well as *real at the same time*.[52] But furthermore, he points out that these definitions denote the reason (*ratio*) or cause (*causa*) of the figure in question.[53] He labels them as *causal definitions* (*definitiones causales*), that is, genetic definitions.[54] Prior to Blancanus, *definitio causalis* was a definition of an attribute as an equivalent to the real definition of the subject term. Blancanus' use of the term *definitio causalis* is beyond the scope of its traditional use, insofar as he accentuates the constructive function of such causal definitions. His example is the definition of a square: he takes it to be the definition that designates the cause for being a square. In many places, Blancanus blurs the distinction between the definition and the construction of a figure.

By assuming causal definitions, Blancanus introduces causes into geometry and into geometrical proofs. With his peculiar view of geometric definitions and constructions, Blancanus undermines the two major presuppositions of the critical objections. On the one hand, he calls into question the distinction between principles of Being and epistemological principles. On the other, the distinction between the essence of a geometric figure and its relations to other figures becomes debatable. For example, some critics object that the demonstration of the equilateral triangle proceeds from the definition of the circle, which is not part of the essence of such a triangle. In contrast to this, Blancanus regards the whole construction as part of the concept of the figure.[55]

With this in mind, the Being of geometrical figures *is* their construction, and Blancanus subverts the two distinctions. Within Blancanus' theory, both distinctions cannot be applied meaningfully to mathematical objects because the geometric figures depend less on abstraction but more on definition and construction. In this perspective, principles of Being and epistemological principles coincide in geometry, insofar as there are no geometrical figures beyond their construction.

With his 'rescue' of the Aristotelian theory of proof, Blancanus gets into trouble. Like the critics, he insists on Aristotelian abstraction as the source of geometrical figures.[56] Yet, Aristotelian abstraction is not entirely compatible with Blancanus' concepts of definition and of construction.[57] *Either* the geometrical proofs are created by abstraction from experienced objects *or* they are constituted by causal definitions and geometrical constructions. It seems to be that Blancanus just needs the abstraction for the creation of the most basic elements like points and lines, while the more complex figures like squares or triangles depend on their definitions. In legitimating the *subjective* certainty of mathematics, Blancanus subverts the justification of its *objective* certainty. The consequence is that his rescue of the Aristotelian view of mathematics fragments precisely the Aristotelian view of mathematics.

4. A New Concept of Geometrical Proof

Such a level of detail in differences implies significant common ground. All participants of the initial debate recognize the Aristotelian scientific theory as the norm. The characteristics of mathematical proofs are only recognizable against the background of different kinds of proofs and "proof theories." The starting point for the debate is the discovery that the Aristotelian theory of proof and Euclid's geometric demonstrations are incompatible. However, no participant in the debate explicitly rejects the Aristotelian theory of proof.

Considering the Aristotelian framework of the initial debate, one can dismiss it as a purely scholastic one. But that does not mean that it has no relevance for the further development of mathematical theory. This debate is where the foundations for the acceptance of a relational understanding of geometry (4.1) were laid. Certainly, some questions discussed in the *Quaestio de certitudine* lose their relevance when Aristotelian logic is abandoned as "the language of science."[58] This mainly pertains to the question of whether *demonstrationes potissimae* are used in mathematics. And yet, some non-Aristotelian scholars restage the debate in the 17th century (4.2). These scholars, indeed, abandoned some preconditions of the initial debate, and therefore transformed its initial question. They changed it even more, by adopting the relational understanding of geometrical proof. This shift from causes to relations fits into a general shift to structures in mathematical thinking which I want to outline as a last point (4.3).

4.1. The Relational Understanding of Geometrical Proof

Within the debate, a new concept of geometrical proof emerges, distinguished by two characteristics.

1) First of all, the arguments of the critics as well as of the defenders imply *a growing degree of flexibility towards geometry's ontological foundations*. The critics emphasize that geometrical proofs are just based on epistemological principles, instead of principles of Being. Defenders like Blancanus blur the distinction between epistemological principles and principles of Being. In doing so, Blancanus rejects Aristotelian abstraction as *the* ontological foundation of mathematics. By breaking away from the ontological foundation of mathematics, he opens up the possibility of applying

mathematics to non-mathematical contexts, whereas within the Aristotelian framework, scientific disciplines are separated by their different and unique subject areas. Only the mixed sciences are mathematized since they are subordinated to mathematical disciplines; optics, for example, is subordinated to geometry.[59] But in Aristotelian understanding, the philosophy of nature is not mathematized.

2) Furthermore, critics like Piccolomini and Smiglecius are not satisfied with the mere statement that geometrical proofs do not meet the requirements of an Aristotelian proof. In addition, they seek to explain how mathematical proofs work, and in the process, they emphasize the role of internal relationships within geometry itself: Geometrical proofs are not based on a hierarchy of causes. The proofs in geometry argue primarily on the basis of the relationships between the different figures. According to this, geometrical proofs are based on relations and coherence. Such a view does not fit into the Aristotelian framework, insofar as geometric proofs in an Aristotelian sense have to proceed from the essence of the subject (as its unique and proximate cause). This objection from the critics presupposes the distinction between the essence of a geometrical figure and its relation to other figures. Blancanus wants to rescue Aristotelian proof theory by blurring this distinction. Finally, this is where a concept of geometrical proof which focuses on the internal structure of geometry and the practice of geometrical constructions emerges.

4.2. The Revival of the Debate

When scholars such as Pierre Gassendi, John Wallis, Thomas Hobbes and Isaac Barrow take up the debate again in the 17th century,[60] "it is mathematics that is the paradigm of science and its reasoning the paradigm of scientific demonstrations."[61] Given this fundamental change, the debate no longer has any interest in rescuing Aristotelianism. Gassendi and Barrow explicitly discuss the certainty of mathematics in the context of scepticism.[62] Wallis, Hobbes and Barrow modify the Aristotelian conditions for demonstrations in order to rescue the certainty of mathematics and its status as a science. Compared to the positions in the initial debate, the strategies of their arguments bear some similarity to Blancanus' arguments, especially when they conceive of the definitions and the constructions themselves as the cause of the figures.[63] We can take Hobbes as an example. Despite his criticism of Aristotelian philosophy, he abides by the Aristotelian idea that knowledge is causal knowledge. Indeed, he amalgamates the concept of causal knowledge with that of *maker's knowledge*.

> [T]he science of every subject is derived from a precognition of the causes, generation and construction of the same; and consequently where the causes are known, there is place for demonstration, but not where the causes are to seek for. Geometry therefore is demonstrable, for the lines and figures which we reason are drawn and described by ourselves, and civil philosophy is demonstrable, because we make the commonwealth ourselves. But because of natural bodies we know not the construction, but seek it from the effects, there lies no

demonstration of what the causes be we seek for, but only of what they may be.[64]

Obviously, Hobbes' conception of causal knowledge does not really match the Aristotelian conception. By combining causal and maker's knowledge, he subverts the distinction between epistemological principles and principles of Being. By doing this, he puts the emphasis on the construction of the figure – much like Blancanus. Like the other proponents of the debate's second active period, Hobbes does indeed reject most of the Aristotelian framework.

4.3. The General Shift to Structures in Mathematical Thinking

Both the initial debate and its revival use a concept of geometrical proof that focuses on the internal structure of geometry and the practice of geometrical construction. Such an understanding of geometrical proof is in tune with general developments in the mathematics of the early modern period.

Parallel to the debate in geometry, a sea change took place in arithmetic and algebra, instigated by the introduction of the decimal number system.[65] Arithmetic broke away from the classical concept of numbers, which is oriented towards counting. The new understanding of numbers regarded them as constituted by the structure of the (positional) number system. The system generates the mathematical entities insofar as the symbols are relationally defined. In 1591, François Viète published his *In artem analyticam Isagoge*, in which he designs his *algebra speciosa*,[66] according to which, the solution of an equation consists of formal transformations, which are based on transformation rules and the relations between the symbols. This process inherently disregards the referents of the symbols. Viète enables general solution methods with uninterpreted symbols. Such methods do not depend on experienced objects. Instead, the interpretation of the symbols depends on the system within which we operate.[67] Such a *relationalism* enables pure *formal* or *syntactical* reasoning, which is the characteristic of algebraic and arithmetic reasoning in modern times.[68]

In contrast to this, the type of proof used in geometry is *content-based* or *semantic*.[69] The axiomatic proofs in Euclid's geometry proceed from definitions, axioms and already proven theorems. Geometric proofs are related to the geometric figures and their spatial features. An (exaggeratedly) strict Aristotelian would regard geometry as a set of unrelated figures.

However, the participants of the debate concerning mathematical certainty put the emphasis much more on the role of the relations between the figures and on the construction of them with respect to their definitions. This presupposes a more flexible stance towards geometry's ontological foundation. In doing this, they bring geometric proof closer to arithmetic proof. They lay the foundations for the *algebraisation* of geometry. More precisely, the shift from causes to relations creates an intellectual climate which enables the *acceptance* of such an algebraisation. In this sense, a relational picture of geometry is a precondition of Descartes' analytical geometry and, in a way, of Leibniz' calculus.

Within Descartes' analytic geometry, the criterion for mathematical existence is not abstraction from experience but constructability. Descartes transfers arithmetics and algebra onto geometry. According to his understanding of geometry, its subject area is based on the construction of figures, which he reduces to arithmetic operations.[70] This program is grounded in the relations between the lines of a figure. The relationalism inspired by the algebra is the presupposition of Descartes' epistemology and metaphysics, insofar as he, in his *Regulae*, understands the being rather than as substance by its relation to others.[71] As Schulthess points out:

> If you want to make an unknown known in the algebra, i.e. in the case of equation systems, you will have to use the method of relations (*habitudo*). That is to express the unknown by its relations to the known or given. This method based on mathematics is the core of the concept of the mathesis universalis in Descartes' *Regulae*.[72]

Leibniz goes one step further than Descartes when he understands calculus as something that establishes relationships by converting formulas. In doing this, Leibniz reduces content-based reasoning to a semiotic-syntactic notion of reasoning.[73]

And this is the sense in which the *Quaestio de certitudine mathematicarum* enabled the new developments in mathematics previously alluded to. Although all participants of the debate abide by the tenets of Aristotelianism, the debate develops its own dynamics: arguments and concepts emerge that finally challenge Aristotelianism, particularly the idea of abstraction and its implications.

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[1] Hume, D., *Philosophical Essays concerning Human Understanding* (London: Printed for A. Millar, 1748), 47 f.

[2] Serene, E., "Demonstrative Science", in *The Cambridge History of Later Medieval Philosophy: From the Rediscovery of Aristotle to the Disintegration of Scholasticism 1100-1600*, eds. N. Kretzmann, A. Kenny and J. Pinborg (Cambridge: Cambridge University Press, 1982), 496-517, here 497. Cf. Aristotle, *Posterior Analytics*, I.2 .

[3] Mancosu, P., "Explanation in Mathematics", chap. 2, *The Stanford Encyclopedia of Philosophy* (Summer 2011 Edition), ed. E.N. Zalta, [Online] Available via:http://plato.stanford.edu/archives/sum2011/entries/mathematics-explanation cited 25.09.2011.

[4] On the reception of the Aristotelian concept of science by medieval scholars, see Serene, E. (1982); Nadler, S., "Doctrines of Explanation in Late Scholasticism and in the Mechanical Philosophy", in *The Cambridge History of Seventeenth-Century Philosophy*, eds. D. Garber and M. Ayers (Cambridge: Cambridge University Press, 1998), 2 vols., vol. I, 513-552; and Longeway, J., "Medieval Theories of Demonstration", *The Stanford Encyclopedia of Philosophy* (Spring 2009 Edition), ed. E.N. Zalta, [Online] Available via http://plato.stanford.edu/archives/spr2009/entries/demonstration-medieval cited 25.09.2011. On the role of Aristotle in Renaissance philosophy, see Schmitt, C.B., *Aristotle and the*

Renaissance (Cambridge, Mass: Harvard University Press, 1983). Concerning competing traditions of mathematical methods (especially in the early modern times) see Gilbert, N. W., *Renaissance Concepts of Methods* (New York/London: Columbia University Press, 1960); Schüling, H., *Die Geschichte der axiomatischen Methode im 16. und beginnenden 17. Jahrhundert: Wandlung der Wissenschaftsauffassung* (Hildesheim/New York: Olms, 1969), chap. 1-6; Engfer, H.-J., *Philosophie als Analysis: Studien zur Entwicklung philosophischer Analysekonzeptionen unter dem Einfluß mathematischer Methodenmodelle im 17. und frühen 18. Jahrhundert* (Stuttgart/Bad Cannstatt: Frommann-Holzboog, 1982) and Schulthess, P., "Die philosophische Reflexion auf die Methode", in *Die Philosophie des 17. Jahrhunderts*, ed. J.P. Schobinger (Basel: Schwabe, 1998), 3 vols., vol. I: Allgemeine Themen. Iberische Halbinsel. Italien, 62-120, here esp. 68 ff. and 77 ff.

[5] See Trudeau, R.J., *The non-Euclidean Revolution* (Basel/Boston/Berlin: Birkhäuser, 1986), chapter "The Psychological Impossibility of Non-Euclidean Geometry", 159.

[6] See Trudeau, R.J. (1986), 155.

[7] Lambert acknowledges that there are plausible interpretations of a geometry without the postulate of parallels. See Lambert, J.H., "Theorie der Parallelinien" [posthumously published, 1786], in *Die Theorie der Parallellinien von Euklid bis auf Gauss: Eine Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometrie*, eds. P. Stäckel and F. Engel (Leipzig: Teubner, 1895), 152-208; see also 144 ff.

[8] On various concepts of science during the scientific revolution, see McMullin, E., "Conceptions of Science in the Scientific Revolution", in *Reappraisals of the Scientific Revolution*. eds. D.C. Lindberg and R.S. Westman (Cambridge: Cambridge University Press, 1990), 27-92.

[9] See Crombie, A.C., "Science and the Arts in the Renaissance: The Search for Truth and Certainty, Old and New", *History of Science* 18 (1980): 233-246, here 234; and Schulthess, P. (1998), 84.

[10] Concerning the influence of the school of Padua on the 17th century scientists, see Randall, J.H., "The Development of Scientific Method in the School of Padua", *Journal of the History of Ideas* 1, 2 (1940): 177-206; and Edwards, W.F., "Paduan Aristotelianism and the Origin of Modern Theories of Method", in *Aristotelismo Veneto e Scienza Moderna*, ed. L. Olivieri, (Padua: Antenore, 1983), 2 vols., vol. I, 206-220.

[11] Cf. for example Alessandro Piccolomini's *Commentarium de certitudine mathematicarum disciplinarum* in his *Alexandri Piccolominei In mechanicas quaestiones Aristotelis, paraphrasis paulo quidem plenior. Eiusdem commentarium de certitudine mathematicarum disciplinarum: in quo, de resolutione, diffinitione& demonstratione: necnon de materia, & in fine logicae facultatis, quamplura continentur ad rem ipsam, tum mathematicam, tum logicam, maxime pertinentia* (Venice: Apud Traianum Curtium, 1565), chap. 10; cf. also Schüling, H. (1969), 42 f. Piccolomini's *Commentarium de certitudine mathematicarum disciplinarum* was first published in 1547 (Rome: Asulanum); here and in the following, I use the second edition from 1565, quoted as Piccolomini, A., *Commentarium de certitudine mathematicarum*.)

[12] Gilbert, N.W. (1960), 89.

[13] Schüling, H. (1969), 41 ff.

[14] On the distinction between objective and subjective certainty, see Schulthess, P. (1998), 84 f.

[15] Aquinas, T., *Expositio super librum Boethii de Trinitate*, ed. B. Decker (Leiden: Brill, 1959), quaestio vi, articulus 1, esp. pp. 208 ff.

[16] Aristotle, *Posterior Analytics*, I.2 and *Nicomachean Ethics* VI.3, 1139b 18 ff.

[17] Aristotle, *Posterior Analytics*, I.14.

[18] For example, Blancanus, J. [Biancani, Giuseppe], *Sphaera mundi seu Cosmographia demonstratiua, ac facili methodo tradita: in qua totius mundi fabrica, una cum novis, Tychonis, Kepleri, Galilaei, aliorumque astronomorum adinuentis continetur. Accessere I. Breuis introductio ad Geographiam. II. Apparatus ad*

Mathematicarum studium. III. Echometria, id est Geometrica traditio de Echo (Bologna: Bonomij, 1620), 406-408. A few years back, he defended the thesis that some geometrical proofs are *demonstrationes potissimae*; see Blancanus, J. [Biancani, Giuseppe], *De mathematicarum natura dissertatio* (Bologna: Apud Bartholomaeum Cochium, 1615).

[19] On the conjunction of the *demonstratio quia* and the *demonstratio propter quid* with other models of methods cf. Schulthess, P. (1998), 80.

[20] Aristotle, *Posterior Analytics*, I.13; for the following reconstruction, see Jardine, N., "Epistemology of the Sciences", in *The Cambridge History of Renaissance Philosophy*, eds. C.B. Schmitt, Q. Skinner, E. Kessler and J. Kraye (Cambridge: Cambridge University Press, 1988), pp. 685-711, here 686 f.

[21] Piccolomini, A., *Commentarium de certitudine mathematicarum*, 77v.

[22] Averroes/Aristotle, *Aristotelis opera cum Averrois commentariis*, vol. I, part 2a: *Aristotelis Stagiritae posterium resolutoriorum libri duo. Cum Averrois Cordubensis magnis commentariis* (Venice: Junctas, 1562), 208 ff., esp. 208v and vol. IV: *Aristotelis de physico auditu libri octo. Cum Averrois Cordubensis variis in eosdem commentariis*, 4r and v. Among the participants of the *Quaestio de certitudine*, Hieronymus Balduinus and Simon Simonius strictly ally to Averroes. For further details, see Balduinus, H., *Expositio in libellum Porphyrii de quinque vocibus* (Venice: Pryphuis, 1562), 222r ff., esp. 223v, and Simonius, S., *Antischegkianorum liber unus* (Basel: editor not mentioned, 1570), 304 ff. Concerning Averroes and his adaption in the debate, see Schüling, H. (1969), 44 f. and Cozzoli, D., "Alessandro Piccolomini and the Certitude of Mathematics", *History and Philosophy of Logic* 28 (2007): 151-171, here 157 ff.

[23] Piccolomini argues that at least one of the premises must be fundamental (*Commentarium de certitudine mathematicarum*, 79r f.).

[24] Usually, they conceive of its middle term as the formal cause. See for example, Balduinus, H., *Expositio in libellum Porphyrii*, 222r ff., esp. 223v and Schegk, J. *Antisimonius* (Tübingen: [Gruppenbach], 1573), 277 ff.

[25] See for example, Averroes/Aristotle, *Posterium resolutoriorum libri duo,* 208 ff., esp. 208v; Averroes/Aristotle, P*hysico auditu libri octo,* 4r and v; Jardine, N. (1988), Cozzoli, D. (2007), and Mancosu, P., "Aristotelian Logic and Euclidean Mathematics: Seventeenth-Century Developments of the *Quaestio de Certitudine Mathematicarum*", *Studies in History and Philosophy of Science* 23, 2 (1992): 241-265.

[26] Piccolomini, A., *Commentarium de certitudine mathematicarum*,79v; Schegk, J., *Antisimonius*, 277 ff.; Hobbes, T., "Examinatio et emendatio mathematicae hodiernae, qualis explicatur in libris Johannis Wallisii distributa in sex dialogos" [1660], in *Thomae Hobbes Malmesburiensis Opera philosophica quae latine scripsit omnia*, ed. W. Molesworth (London: Bohn,1839-1845), 5 vols., vol. IV, 1-232, esp. 38 and Wallis, J., *Mathesis Universalis* (Oxford: Lichfield, 1657), chap. 3. Wallis doubts in this work (10 f.) whether there is a *demonstratio potissima* in any science.

[27] Proclus already questions if all of Euclid's proofs meet Aristotle's requirements of demonstration. For further details see Proclus, *A Commentary on the First Book of Euclid's Elements*, ed. Glenn R. Morrow (Princeton: Princeton University Press, 1970), 161 f.

[28] On aspects of this debate, see Schüling, H. (1969), chap. 9, Gilbert, N.W. (1960), 90 f.; Jardine, N. (1988), 693-697; Mancosu, P. (1992); Mancosu, P., *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century* (New York and Oxford: Oxford University Press, 1996), 8-33 and Schulthess, P. (1998), 84.

[29] Balduinus, H., *Expositio in libellum Porphyrii*, 222r ff., esp. 223v; Schegk, J., *Antisimonius*, 277 ff.

[30] On Piccolomini's discussion of the mathematical certainty, See Schüling, H. (1969), 45 f.; Giacobbe, G. C., "Il *Commentarium de Certitudine Mathematicarum Disciplinarum* di Alessandro Piccolomini", *Physis: Rivista Internationale di Storia della Scienza* 14, 2 (1972): 162-193; Jardine, N.

(1988), 693-697; Mancosu, P. (1992), 243 f. and Mancosu, P. (1996), 12 f. and Cozzoli, D. (2007).

[31] Piccolomini, A., *Commentarium de certitudine mathematicarum*, 84v ff.

[32] Piccolomini, A., *Commentarium de certitudine mathematicarum*, 100v.

[33] Piccolomini, A., *Commentarium de certitudine mathematicarum*, 103v ff.

[34] Cf. Simonius, S., *Antischegkianorum*, 304 and 310.

[35] For further details, see Piccolomini, A., *Commentarium de certitudine mathematicarum*, 96r ff. and 100v f.

[36] See Piccolomini, A., *Commentarium de certitudine mathematicarum*, chap. 7 and 101r f.

[37] Piccolomini argues for the thesis that the middle term must signify the definition of the property and cannot be the definition of the subject itself (*Commentarium de certitudine mathematicarum*, 81r ff. and 88v ff). But, he disproves both conceptions since many of his predecessors regard the middle term as the definition of the subject.

[38] See Piccolomini, A., *Commentarium de certitudine mathematicarum*, 102r; concerning his reconstruction of Euclid I.32, see *Commentarium de certitudine mathematicarum*, chap. 9. Euclid I.32 and Euclid I.1 are the two proofs of Euclid which run like a golden thread through the debate; see Euclid, *The Thirteen Books of Euclid's Elements*, ed. T. L. Heath, (New York: Dover, 1956), 3 vols., vol. I, 316 ff. and 241 ff. Since most participants of the debate usually use these proofs of Euclid as paradigm (cf. Mancosu, P. (1992), 262) I will use these two proofs to illustrate their concept of geometrical proof.

[39] Proclus, *A Commentary on the First Book of Euclid's Elements*, 161 f. and also Schüling, H. (1969), 43.

[40] Piccolomini regards axioms and definitions as fundamental propositions. For this point and the following see *Commentarium de certitudine mathematicarum*, 83r f. and102r ff.

[41] "[…] eadem conclusio in Mathematicis potest demonstrare per pluras, & diversas praemissas." (Piccolomini, A., *Commentarium de certitudine mathematicarum*, 103r; cf. also ibid., 105r f.). Piccolomini attributes this thesis to Themisticus.

[42] Piccolomini conceives the mathematical entities as abstracted quantities (*quantitates ipsae abstractae*); cf. *Commentarium de certitudine mathematicarum*, 94v. For his constructive approach to the mathematical certainty, cf. *Commentarium de certitudine mathematicarum*, chap. 12; see also Carugo, A., "Guiseppe Moleto: Mathematics and the Aristotelian Theory of Science at Padua in the Second Half of the 16th Century", in Olivieri, L. (1983), vol. I, 509-517, here 511 f.

[43] Pererius, B. [Pereira, Benito] *De communibus omnium rerum naturalium principiis & affectionibus, libri quindecim* (Cologne: Zetzner, 1603), esp. 116-122 (first published 1579: Pereriu, B [Pereiro, Benito], *De communibus omnium rerum naturalium principiis et Affectionibus, Libri Quindecim* […] (Paris: Michael Sonnius, 1579)); Conimbricenses, *Commentarii Collegii Conimbricensis e Societate Iesu: In universam dialecticam Aristotelis Stagiritæ* (Cologne: Gualterius, 1611), col. 501-507 and Smiglecius, M., *Logicae. Pars altera. Ea omnia, quae ad secundam & tertiam operationem intellectus pertinent comprehendens* (Ingolstadt: Typographeo Ederiano, 1618), 300-308.

[44] Pererius, B., *De communibus omnium rerum naturalium principiis &affectionibus*, 117 f., Conimbricensis, *In universam dialecticam Aristotelis*, 506; and Smiglecius, M., *Logicae*, 305.

[45] Euclid I.1 (Euclid, *Elements*, 241 ff.). Cf., for example, Piccolomini, A., *Commentarium de certitudine mathematicarum*, chap. 10.

[46] "Quia in demonstratione potissima, causa proprietatis est essentia subiecti, a qua proprietas illa oritur: at in Mathematicis non probantur prorietates ex essentia subiecti, sed ex habitudine ad aliam figuram: Ergo non probantur per veram causam essendi." (Smiglecius, M., *Logicae*, 306)

[47] "Mea opinio est, Mathematicas disciplinas non esse proprie scientas […]. Scire es rem per causam cognoscere propter quam res est; & scientia est demonstrationis effectus […]." (Pererius, B., *De communibus omnium rerum naturalium principiis et affectionibus*, 40) see also Schüling, H. (1969), 47 ff. and Giacobbe, G.C., "Un Gesuita Progressista nella *Quaestio de Certitudine Mathematicarum* Rinascimentale: Benito Pereyra", *Physis: Rivista Internationale di Storia della Scienza* 19, 2 (1977): 51-86.

[48] Barocius, F. [Barozzi, Francesco], *Opusculum in quo vna oratio & duae qu[a]estiones, altera de certitudine & altera de medietate mathematicarum continentur* (Padua: E.G.P. [i.e. Gratiosus Percacinus], 1560) and Blancanus, J., *De mathematicarum natura dissertatio*, esp. 11; on the different kinds of mathematical proofs Blancanus assumes, see *De mathematicarum natura dissertatio*, 10. See also Giacobbe, G.C., "Francesco Barozzi e la *Quaestio de Certitudine Mathematicarum*", *Physis. Rivista Internationale di Storia della Scienza* 14, 4 (1972): 357-374 and Giacobbe, G.C., "Epigoni nel seicento della *Quaestio de Certitudine Mathematicarum*: Guiseppe Biancani", *Physis. Rivista Internationale di Storia della Scienza* 18, 1 (1976): 5-40.

[49] Blancanus, J., *De mathematicarum natura dissertatio*, 11 f. and esp. 14 and Barocius, F., *Opusculum*, 21v f. Barocius maintains that there are two types of definition used as the middle term in geometrical proofs, namely the material definition (*definitio materialis*) and the formal definition (*definitio formalis*). He argues that some geometrical proofs infer from causes, insofar as the definitions explain their causes. See also Barocius, *Opusculum*, 24r and Schüling, H. (1969), 52 f.

[50] Blancanus, J., *De mathematicarum natura dissertatio*, 14 f. and also Schüling, H. (1969), 53 ff.

[51] Blancanus, J., *De mathematicarum natura dissertatio*, 16 f. and also Schüling, H. (1969), 53 ff.

[52] Blancanus, J., *De mathematicarum natura dissertatio*, 7 f. In contrast, Piccolomini regards the definitions in mathematics as nominal definitions and denies the assumption that these definitions could be real definitions; cf. *Commentarium de certitudine mathematicarum*, 84r.

[53] Blancanus, J., *De mathematicarum natura dissertatio*, 7-9 and 16.

[54] Blancanus, J., *De mathematicarum natura dissertatio*, 9.

[55] Blancanus, J., *De mathematicarum natura dissertatio*, 9.

[56] Blancanus, J., *De mathematicarum natura dissertatio*, 5-7. "Quantitas igitur abstracta a materia sensibili […] considerari solet." (*De mathematicarum natura dissertatio*, 5). He maintains that geometric figures are created by combining the abstract quantities, wherefore the mathematical entities have just a mental reality (cf. *De mathematicarum natura dissertatio*, 6 f.).

[57] Blancanus complicates this idea by combining it with the neoplatonic idea that the originals of the geometric figures (pre)exist in a divine mind; cf. Blancanus, J., *De mathematicarum natura dissertatio*, 7.

[58] Mancosu, P. (1992), 255.

[59] Aristotle, *Posterior Analytics*, I.13.

[60] Gassendi, P., "Exercitationes Paradoxicae adversus Aristoteleos" [1624], in *Petri Gassendi Opera omnia*, (Lyon: Anisson & Devenet, 1658), 5 vols., vol. III, 92-193, here excercitatio 6; Wallis, J., *Mathesis Universalis*, chap. 3, Hobbes, T., "Examinatio et emendatio mathematicae hodiernae", esp. 35-43 and Barrow, I., *The Usefulness of Mathematical Learning* [English translation of Barrow's *Lectiones*, held 1665, published posthumously 1683], trans. J. Kirkby (London: Austen, 1734), esp. lectures 5 and 6. Cf. also Pycior, H., "Mathematics and Philosophy: Wallis, Hobbes, Barrow and Berkeley", *Journal of the History of Ideas* 48, 2 (1987): 265-286, Mancosu, P. (1992), and Stewart, I., "Mathematics and Philosophy: Barrow and Proclus", *Dionysius* 18 (2000): 151-182.

[61] Mancosu, P. (1992), 255.

[62] Gassendi, P., "Exercitationes Paradoxicae adversus Aristoteleos", excercitatio 6; Barrow, I., *The Usefulness of Mathematical Learning*, esp. lecture 5; also Mancosu, P. (1992), 258 ff., esp. 265.
[63] Wallis, J., *Mathesis Universalis*, 11 and Barrow, I., *The Usefulness of Mathematical Learning*, 78 ff. Barrow regards some of the definitions as analytical statements and considers the internal relation of mathematical terms as formal cause (*The Usefulness of Mathematical Learning*, 88).
[64] Hobbes, T., "Six Lessons of the Professors of the Mathematics, one of Geometry, the other of Astronomy" [1656], in *The English Works of Thomas Hobbes of Malmesbury*, ed. W. Molesworth (London: Bohn, 1839-1845), 11 vols., vol. VII, 181-356, here 184. Concerning the maker's knowledge tradition, see Pérez-Ramos, A., *Francis Bacon's Idea of Science and the Maker's Knowledge Tradition* (Oxford: Clarendon, 1988).
[65] Krämer, S., *Symbolische Maschinen: Die Idee der Formalisierung in geschichtlichem Abriß* (Darmstadt: Wissenschaftliche Buchgesellschaft, 1988), 54 ff. and Krämer, S., *Berechenbare Vernunft: Kalkül und Rationalismus im 17. Jahrhundert* (Berlin and New York: de Gruyter, 1991), 88 ff.
[66] Cf. Viète, F., *In artem analyticam Isagoge* (Tours: Mettayer, 1591) and also Krämer, S. (1991), 124 ff.
[67] See Stevin, S., *L'Arithmetique* (Leyden: Plantin, 1585) and also Krämer, S. (1988), 59 ff. and Krämer, S. (1991),esp. 143.
[68] On the formal or syntactical reasoning in algebra distinguished from the contentual or semantical reasoning in geometry, cf. Schulthess, P. (1998), 96 and 105.
[69] Schulthess, P. (1998), 96 and 105.
[70] Cf. his geometry in Descartes' *Discours*; see Descartes, R., *Discours de la méthode pour bien conduire sa raison,& chercher la verité dans les sciences. Plus la Dioptrique. Les Meteores. Et la Geometrie. Qui sont des essais de cette Methode* (Leyden: Maire, 1637); and also Krämer, S. (1991), 151.
[71] Cf. Descartes, R., *Regulae ad directionem ingenii / Regeln zur Ausrichtung der Erkenntniskraft* [1628].ed. H. Springmeyer, L. Gäbe und H.G. Zekl (Hamburg: Meiner, 1973), 381 and 383; see also Schulthess, P. (1998), 93.
[72] "Will man in der Algebra, also bei Gleichungssystemen, eine Unbekannte bekannt machen, so muss man die Methode der Beziehungen (*habitudo*) anwenden, d.h. das Unbekannte durch Beziehungen zu Bekanntem, Gegebenem ausdrücken. Diese an den Beziehungen in der Mathematik orientierte Methode steht auch im Vordergrund des Konzepts einer mathesis universalis in den *Regulae*." (Schulthess, P. (1998), 94.). On the *mathesis universalis* and the *mos geometricus* cf. Arndt, H.W., *Methodo scientifica pertractatum: Mos geometricus und Kalkülbegriff in der philosophischen Theorienbildung des 17. und 18. Jahrhunderts* (Berlin/New York: de Gruyter, 1971).
[73] Leibniz, G.W., "De ortu, progressu et natura algebrae, nonnullisque aliorum et propriis circa eam inventis", in *Leibnizens mathematische Schriften*, ed. C.I. Gerhardt (Berlin: Asher; Halle: Schmidt, 1848-1863), 7 vols., vol. VII, 203-216, here 206 and also Schulthess, P. (1998), 108.