INSTANCES OF DESCARTES'S EARLY PROJECTIONISM: THE CORRESPONDENCE WITH FERRIER

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Abstract. This paper investigates Descartes's description of a lens-cutting machine set forth in his early correspondence with the artisan Jean Ferrier. I argue that this episode of Descartes's mixed-mathematical practice goes beyond the traditional Aristotelian model of subalternation of sciences in two ways: (i) it conceives optics as not being bound *only* to the principles of geometry – as it should be the case for Aristotelian mixed-mathematics, and (ii) it allows for conjoining different types of mixed-mathematics in a way in which certainty is preserved. The knowledge of each of them is seen as transferable to the other. In turn, these particularities are better explained if we take Descartes to have a view on mixed-mathematics which is inspired by Proclus's *Commentary on the First Book of Euclid's Elements*.

Keywords: Descartes, Ferrier, Proclus, division of sciences, mixed-mathematics, dioptrics

Introduction

René Descartes's 1637 *Dioptrics* ends with a chapter on designing the construction of a lens-cutting machine. The method of cutting lenses was an older pursuit of his, first tackled in 1629, in the correspondence with Jean Ferrier. This paper argues that Descartes's early ambition to devise a lens-cutting machine goes beyond the traditional Aristotelian model of subalternation of sciences in two ways: (i) it conceives a mixed-mathematical discipline – i. e. optics – as not being bound *only* to the principles of the higher science – i. e. geometry – and (ii) it allows for conjoining different types of mixed-mathematics in a way in which certainty is preserved. The knowledge of each of them is regarded as transferable to the other, and this is granted by Descartes's construal of machines as idealized heuristic tools.

I suggest that the particularities of Descartes's mathematical practice of describing a lens-cutting machine should not be accommodated within the traditional Aristotelian model of sciences. Instead, Descartes's design of the machine better fits a Proclean interpretation of the place which mixed-mathematics occupies in the general

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hierarchy of knowledge. Regarding (i), the middle ground ontological status that Proclus ascribes to mathematical objects and mathematical reasoning provides a satisfactory framework in which mixed-mathematical disciplines can, in their turn, furnish principles to natural philosophy. As for (ii), the projectionist Proclean view set forth in *The Commentary on the First Book of Euclid's Elements* entails that the knowledge obtained from distinct branches of mixed-mathematics may permeate each other regardless of the nature of the sensible objects which they deal with.

Circumscribing Descartes's early mixed-mathematical practice within one of the two models of science might seem an odd endeavor. First, why are there only two options – the Aristotelian and the Proclean – for accessing Descartes's understanding of optics as a mixed pursuit? One reason is that both Descartes's education and the ongoing debates at that time suggest that Descartes was acquainted with the two models of the division of sciences. The *quadrivium* taught at the Jesuit college of La Flèche was to a great extent Aristotelian[1], while Francesco Barozzi's 1560 translation of Proclus's *Commentary* occasioned the Quaestio de certitudine mathematicarum, a debate about the status of mathematical knowledge. Another reason is that the Aristotelian hierarchy of sciences was the background framework against which mixed-mathematical studies were assessed. Proclus's division of sciences could have been regarded – as Cristopher Clavius himself did – as an alternative frame of mixed-mathematical knowledge.

Second, why should Descartes be bound to either one of these models? After all, he boldly asserts that "my whole physics is nothing but geometry"[2] and "my whole physics is nothing but mechanics".[3] The reason is that the present case-study is neutral from the point of view of natural philosophy. It just offers a glimpse into the actual practice of Descartes's mixed-mathematics. The episode originates from 1629, being one of the first attempts of Descartes in dioptrics. This predates his natural philosophy studies, as Descartes started working on *Le Monde* in late 1629.[4]

As to the boundaries of a Proclean influence on Descartes, several scholars acknowledge Proclus as a historical source in studying Descartes's mathematics.[5] Sasaki provides a detailed account of how Proclus's *Commentary* influenced the mathematical thought of Clavius.[6] As Clavius designed the whole mathematical curriculum of the *ratio studiorum* in 1599, there are good reasons to suppose that Descartes became acquainted with Proclus's *Commentary* in the college of La Flèche. Schuster argues for a link between Proclus's general mathematics and Descartes mathesis universalis.[7] Hattab makes a compelling case in arguing that several seemingly conflicting passages from Descartes's writings regarding the ontological status of eternal truths can be resolved if we interpret them as following a view derived from Proclus.[8]

Descartes's early mixed-mathematical practice has been labeled as physico-mathematics and seen as appropriating or consuming various devices of practical mathematics into his natural philosophy.[9] Regarding the early optical inquiries, Schuster notes that Descartes wanted "to make and 'show' lenses that would embody his law of refraction, and control an improved telescope", and that his "lens grinding machine was also a physical/mechanical instantiation of the law of refraction".[10] The general account of Descartes's mixed-mathematical practice as a physico-mathematics entails that the resources appropriated from practical mathematics were somehow made translatable for his natural philosophy.

Within the Aristotelian division of sciences, which was still almost ubiquitous in the first half of the seventeenth century, sciences are individuated according to their subject matter – that is, according to the type of objects which they deal with. If their objects are different, their principles cannot be the same. This is the case of the mixed-mathematical sciences – optics, harmonics, mechanics and the later additions: they are subalternated to the higher science of mathematics, and their deductive chains can only work if they 'borrow' principles from it. Subalternation was seen as applying unidirectionally, and this is a crucial trait for any Aristotelian mixed-mathematical knowledge. Given that mathematical objects are essentially different from natural objects, the study of optics has to refer to both types of beings: timeless abstractions and sensible, material objects. Thus, joining mathematical and material knowledge is the foremost problem that needs to be accounted for in regarding a mixed-mathematical practice as belonging to the Aristotelian division of sciences.

How are we to account for Descartes's possibility of translating and appropriating these different forms of knowledge within the general hierarchy of sciences at that time? It may be seen as fitting the traditional Aristotelian division of sciences. Biener introduces the label 'tempered-hylomorphism' to broaden the scope of the early modern concept of subalternation. He argues that the subalternation model of the division of sciences can apply not only to mixed-mathematical disciplines, but can also fit the relationship of physics and metaphysics in the *Principles*.[11] 'Tempered-hylomorphism' should account for the way in which a mixed-mathematical practice treats mathematical and material considerations jointly, whilst not disturbing the various Aristotelian classifications of sciences.

I argue that Descartes's design of a lens-cutting machine described to Ferrier in 1629 illustrates another way in which the place of his mixed-mathematics in the general division of sciences is established. His applied mathematical practice of attaining a geometrical proportion for determining the angle of refraction of a transparent medium is dependent on the material properties of the medium. The design of the machine based on this proportion is seen as finding properties that would ultimately (ii) allow for conjoining of another mixed-mathematical field in a way in which certainty is preserved. The reasoning involved in finding the anaclastic shape of the lenses, in turn, provides principles to natural philosophy, while (i) not being unidirectionally bound to geometry.

All of these features which will be detailed in the third section are better accommodated within a projectionist Proclean way of understanding mathematical objects and the constitutive role that mathematical reasoning plays in mixed sciences. In the next section I will present the base tenets of the Aristotelian division of sciences. While this model was not seen as a rigid one and had its variations, the essential features of this division were constant. I will then compare the Aristotelian model of sciences with Proclus's view on mixed-mathematics and explore the possible consequences that this latter view would have on Descartes's mixed-mathematical practice.

I. A short detour: subalternation, mathematical objects as abstractions and projections

This section will start with a sketch of the Aristotelian way of conceiving mixed-mathematical disciplines. Two features of this view are important for the present argument: the *modus considerandi* that the mixed-mathematician ought to have when studying nature mathematically, and the idea that mathematical objects are abstractions from particular instances. The *modus considerandi* is the result of Aristotle's genera-crossing constraint, while abstractionism regarding mathematical objects bounds optics to geometrical principles.

In an Aristotelian framework, sciences are unified if they share the same principles. For this, the objects of these sciences must belong to the same genus. This is the main problem of the optics and of the other mixed-mathematical sciences, whose objects seem to cross genera. On the one hand, Aristotle states that:

> It is not possible to demonstrate by crossing from another genus; for example, what is geometrical by means of arithmetic... [I]t is clearly impossible, for the extremes and the middles must be from the same genus.[12]

Still, the possibility of genera-crossing is allowed, but very ambiguously: "the genus must be either the same without qualification or somehow the same if the demonstration is going to cross"[13]. The reason why mathematics was, in itself, inadequate for the study of natural phenomena is the implied genera-crossing of its demonstrations about sensible objects. Mixing mathematics with physics is, nevertheless, possible. As any science in the Aristotelian sense is a deductive chain of syllogisms, the premises of some sciences may be proved by others. Whenever this happens, the former is said to "borrow" principles from the latter, thus becoming subalternated to it. Optics uses geometrical principles in this fashion. The subalternation is only unidirectional, as geometry is necessary for the study of optics (i. e. the behavior of the rays of light), while optics is not necessary for geometry. If a certain optical account of refraction would not fit any actual physical medium, then, on the subalternation model, its mathematical apparatus would just reduce to geometry.

It all comes down to the types of objects that each science in the mix refers to. Physical objects – i. e. objects that belong to natural philosophy – have the source of movement and change within them, being inseparable in both thought and reality. For this, they cannot be considered independently from their matter. On the other hand, mathematics is concerned with entities that are timeless, separable in thought, but not in reality. This is why mixed mathematical sciences cannot have unitary principles, and need to "borrow" them from mathematics. The essence of mathematical objects, as conceived by Aristotle, is such that their usage in the study of nature entails genera-crossing. Mathematical objects are abstractions, they are grasped by abstracting away certain features of particular objects – shape rather than weight, volume rather than density, etc. This view was labeled *abstractionism*.[14] The same holds for a mathematical demonstration within pure and mixed-mathematics. It is a collection of propositions which are generalizations from particular instances:

> If the universal is not a thing apart from the particulars, and demonstration instils an opinion that that in virtue of which it demonstrates is some thing, and that this belongs as a sort of natural object among the things there are […] and a demonstration about something there is is better than one about something that is not, and one by which we will not be led into error is better than one by which we will be, and universal demonstration is of this type […] if this is more universal and is less about something there is than the particular demonstration, and instils a false opinion, then the universal will be worse than the particular.[15]

So, the mathematical demonstration within the natural science should only apply to the particular instance which is considered. This is how a principle from the higher science can provide knowledge to the subalternated field: It may never exceed the boundary of the particular instance. It does not matter if the particular is thought of as an individual sensible object or a deductive chain: the properties of the abstracted object are identical with the real natural objects only as long as the reasoning within the mixed-mathematics remains mathematical. This feature is reflected at the methodological level, in the practice of the mixed-mathematician. A certain *modus considerandi* must be used in order to accommodate the mathematical and the natural properties of the subject of study. The *modus considerandi* can just mean that the procedure of investigation is mathematical, while the objects *per se* are natural, belonging to natural philosophy.[16] The *modus considerandi* can also mean that the practitioner should simultaneously regard both the mathematical features and the matter which contains them.[17] Although it embodies the mathematical reasoning, the material *substratum* is only accidentally connected to the mathematical account.

Thus, the mixed-mathematician has to continuously be wary of the accidental character of the matter involved. It is the only proper way in which the mixed-mathematician can jointly consider objects that belong to mathematics and those that belong to natural philosophy. The point is crucial for the present argument. The way in which this *modus considerandi* is conceived imposes a methodological constraint upon the mixed-mathematician. In the case of the optics, the mathematical apparatus may not be (or even stronger, *is not*) bound to the matter of the instance that is studied. If matter is accidental, then the only acceptable type of demonstration of the premises in the deductive chain belongs to the higher science – i. e. geometry. Certainty is thus granted by the legitimacy of geometrical inferences.

Such features of the subalternation model came under attack in the Renaissance, and the discussion continued in the early modern period. But, despite disputes over particular issues of the model, the subalternation model remained the dominant one, such that the general classification of knowledge and hierarchy of sciences followed these lines.[18] At the same time, the genera-crossing constraint in the mixed-mathematical sciences was becoming more and more flexible.[19] A reason for this might be an alternative way of considering mathematics and its role within the large-scale classification of knowledge. It was caused by the rediscovery of Proclus. Simon Grynaeus the Elder published the Greek text in 1533, and it was translated into

Latin by Francesco Barozzi in 1560. Proclus was a Neoplatonic, and his view on mathematical objects, as a synthesis of Plato and Aristotle, is now labeled *projectionism*.[20]

The *projectionist* interpretation of Proclus takes mathematics to be a middle ground between sensible experience and being.[21] Mathematical objects are neither empirical nor pure forms. They are intermediaries. In contrast to sensible objects, mathematical objects are devoid of sensible matter and are not the subject of change.[22] Even so, they are intelligible and provide demonstrations, but they do not have discrete or divisible extension – there are no points without parts or lines without depth. The mind is only reminded of mathematical objects by sense perception, and reasoning of them amounts to unfolding their content with the help of unified intelligence or *dianoia.*[23] *Dianoia*, or the mathematical understanding, is a step by step discursive process of integrating elements of both *nous* (which cannot be grasped discursively) and sense perception (which is imprecise and fragmentary). The faculty of understanding is clear and precise and is dependent upon higher forms of intelligence (*nous*) for providing its principles. The understanding has, thus, a twofold reference. It develops content from the *nous* in a specific ordering while imitating its simplicity and unity. In ordering the received ideas, the understanding relies on a special capacity – the imagination. It is the imagination that presents these ideas as divisible, extended and formed, in an intelligible mathematical matter, space. The imagination does not present pure ideas, but "pictures of them" having the same essence.[24] For instance, a mathematical object like a circle, before being projected into the screen of the imagination, lacks extension, center or circumference. It gets all these properties from the matter which is provided by imagining it. Mueller claims that ascribing this role to the imagination is Proclus's most important novelty, and Claessens points out that this feature was almost completely ignored.[25]

Conceived as a projection, mathematics is a bridge or ladder that prepares the soul for ascending to the higher forms of nous. An important consequence for the present claim is that, as Mueller puts it, "for Proclus, a higher level produces the next lower level and the lower level is a copy of the higher 'in another medium'"[26]. The large-scale Proclean classification of knowledge is different than Aristotle's: the highest form is The One (Being), to which all the souls should transcend. Theology is next, with its knowledge apprehensible in a non-discursive way - through the *nous*. Mathematics occupies the third place, providing discursive knowledge with the aid of the imagination, and physics is last, using ideas provided both by theology and mathematics to form opinions about objects as parts and wholes. However, for Proclus, genera-crossing is not such a big constraint in the case of mixed-mathematics as it is for Aristotle. Proclus writes:

> [Mathematical reasoning] extends from on high all the way down to conclusions in the sense world, where it touches on nature and cooperates with natural science in establishing many of its propositions, just as it rises up from below and nearly joins intellect in apprehending primary principles. In its lowest applications, therefore, it projects all of mechanics, as well as optics and catoptrics, and many other sciences bound up with sensible things and

> operative in them, while as it moves upwards it attains unitary and immaterial insights that enable it to perfect its partial judgements and the knowledge gained through discursive thought, bringing its own genera and species into conformity with those higher realities and exhibiting in its own reasonings the truth about the gods and the science of being.[27]

But how exactly *is* mathematics – within a mixed-mathematical application – able to bring its own apparatus in conformity with the higher realities, while still establishing many propositions of the natural science? The imagination and the understanding are not arbitrary and passive, but driven by the higher forms of being and sense data. The way they are regulated by the higher forms of being provides certainty to mathematics. The principles that are above arithmetic or geometry are generated this way and they not dependent on the type of quantity considered. This is the general mathematics, and its principles are the *limit* and the *unlimited*. The *limit* grounds anything that can be grasped. Any intelligible proposition of mathematics, be it pure or applied, depends on the *limit*. The *unlimited* grants the fecundity of being, the constant possibility of mathematics to be refined, when compared with the particular objects of nature. Principles like equality and inequality, likeness and unlikeness, harmony and disharmony belong to general mathematics. They transcend their field of definition and application, allowing mathematics to:

> [make] contributions of the very greatest value to physical science. It reveals the orderliness of the ratios according to which the universe is constructed and the proportion that binds things together in the cosmos. […][28]

This way of considering the role of mathematics in the study of nature is also manifest in Clavius's works. When he discusses the division between pure and applied mathematics, he states that "some of the mathematical sciences are directed toward the intellect insofar as they are separated from all matter, but others pertain to the senses insofar as they pertain to passive sensible matter."[29] Clavius's division is somehow half-way between Aristotle's and Proclus's. Even if Clavius accepts a clear-cut distinction of pure and applied mathematics, he also places applied mathematics, in a very Proclean vein, within the realm of sensible objects.

For the argument I am making here, two things are important. The aim is to circumscribe Descartes's early optical practice into one of the two models of the division of sciences which I discussed in this section, the Aristotelian one and the Proclean one. First, for Proclus, the common mathematics is not exhausted by the already existing types of procedures and applications which a mixed-mathematics uses in the study of nature. As common mathematics transcends the *nous* and *dianoia*, it may project more mixed mathematical disciplines and more applications onto nature. As for Aristotle, the status of mathematical reasoning within mixed fields – a universal collection of particular instances – binds the addition of more mathematical apparatus to material considerations of natural philosophy. Second, because the nature of mathematical objects is different for Aristotle and Proclus, the constraint of genera-crossing is also different. For Proclus, the *modus considerandi* of the mixed-mathematician need not account for the accidental character of the matter in question. On a charitable reading, this may also hold for Aristotle, but somewhat of a constraint would still remain, as the practitioner has to continuously keep in mind that her/his mixed-mathematics is also *about* the matter that s/he is considering.

II. Optics between mixed mathematics and practice: the case of Descartes and Ferrier

In the seventeenth century, there was a gap between the theoretical mixed-mathematical study of optics and the practice of making good quality lenses. Especially as a result of Galileo's telescopic observations made in 1610, lenses and their properties attracted not only artisans, but mathematicians and philosophers as well.[30] The status of the optics as a mixed-mathematical science was debated in the circle of Descartes. In *La verité des Science*, published in 1625, Mersenne divided the study of optics into three branches. Optics proper, the study of rectilinear propagation of light, catoptrics, the study of reflection and mirrors, and dioptrics, dealing with refraction[31]. Catoptrics did not raise too many problems regarding the shaping of mirrors. The angle of an incident ray is always equal to the angle of the reflected ray. Thigs were more complex in the case of dioptrics. Glass lenses were designed to be spherical by philosophers such as Della Porta, mostly due to the common theory of light propagation, the *perspectiva*.[32]

The prior usage of lenses was diverse: spectacles for correcting human vision were around from the late thirteenth century, and spyglasses were regularly used for navigation and military purposes.[33] Still, telescopes used for astronomical observations raised a number of problems, especially due to philosophers' drive for perfecting them. Even so, as Burnett argues, lens-grinding techniques changed little between 1590 and 1690.[34] Improvements in telescopes – lenses, shape, design – were done in conjunction by mathematical adepts, philosophers and instrument makers. The advances that emerged from philosophers were not determined by mathematical analysis, but by their capacity as telescope makers. It was the actual refinement of crafts accounts for the increase in the resolving power and clarity of telescopes.[35] Subtleties of technique like improving brass pans, polishing materials or more complex lathes made better telescopes, but the principle of production remained the same.

The problems associated with spherical lenses were visual aberrations. Spherical aberration consists in parallel rays that enter the lens but are refracted in different focal points. The focal point of the rays closer to the center of the lens is always farther away than the focal points of the rays closer to the edge of the lens. This results in images that are partly blurred. Chromatic aberration is determined by differential refraction of different wavelengths of light, and made images with areas of unnatural coloring.[36] Surfaces that would be free of spherical aberration were called aplanatic surfaces. Spherical form was believed not to be aplanatic, and the best aplanatic surfaces were considered to be elliptical, hyperbolic or parabolic. An important thing in considering the shape that lenses ought to have is that the anaclastic line – i. e. the shape of the section through the lens – is dependent on the material of the lens. Every transparent medium (air, water, glass, crystal) has a different index of refraction, and thus the anaclastic line has to be established for each material independently.

After 1625, many philosophers became interested in finding out the anaclastic line.[37] Descartes was no exception. To this end, from around 1627, Descartes tried to devise a lens-cutting machine for obtaining lenses with aplanatic properties as precisely as possible. The design plans for the machine are made explicit throughout his correspondence with Ferrier and his 1637 *Dioptrics*. His exchanges with the artisan Ferrier amounted to five letters, in which Descartes was trying to employ the artisan to construct the lens-cutting machine.[38] As it was the fashion, employment would have had to be conducted under Descartes's close supervision.[39] For this, Descartes insisted that Ferrier would have to move to Holland, a thing which Ferrier never did.[40] In the first letter of the exchange, Descartes, with his characteristic optimism, writes:

> Since I left you I have learned many things regarding our lenses, so I thing I can succeed in something which surpasses every known development; and it seems so easy to be accomplished, and so certain that I can hardly doubt the manual labor involved, as I did before"[41]

In the next letters to Ferrier, Descartes gives a detailed description of the envisaged lens-grinding machine. However, in the correspondence, Descartes starts the discussion from the sketch itself, without giving many clues on the conceptual moves he used for the realization of the design plan. This is not the case in the *Dioptrics*. There, Descartes reorganizes the whole design process, with insignificant design changes.

I argue that there are three independent stages that Descartes describes for the machine to cut the lenses by the anaclastic shape in a precise manner. The demarcation between the stages is not merely for presentational purposes, but due to the fact that Descartes was doing different things in each one. The first is a material stage that has to do with the nature of the transparent medium used for the lens. The second is a mathematical stage that involves translating the empirical data onto paper, a "rule of thumb" version of the law of refraction, and a geometric way of tracing the needed hyperbola using only a ruler and a compass. The third stage involves finding a mechanical way to reach the same curve obtained in the second stage, as well as designing the actual machine that can do this job in the simplest possible way. In this section I deal with these stages with the purpose of showing that the knowledge obtained in each one does not refer to objects of the same genus. At the same time, passing on from one stage to the next is envisaged by Descartes as preserving certainty, but this certainty is not only determined by the principles of the higher science, geometry.

1. The material stage

The exchange between Ferrier and Descartes references many practical and artisanal aspects of the construction of the machine. One particular aspect is of interest here: the problem of the "glass triangles". The "glass triangles" are actually prisms, and their only purpose is to establish the index of refraction of the material. Ferrier tells Descartes that he knows the glass triangles can be constructed with any angle of inclination, provided they are made of the same type of glass, but he does not have the means to test it.[42] Descartes replies that the index is measured according to a proportion between certain points.[43] For this, a small practical setup needs to be put into place: a straight and flat plank neither too shiny nor transparent, two small plates, each with a pinhole through which a ray of light should pass, and the glass prism. The small plates and the prism have to be precisely perpendicular on the flat plank. Here is the illustration offered by Descartes:

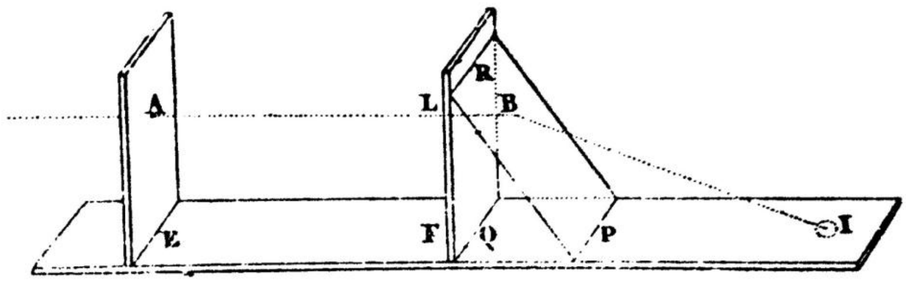


Illustration of the first stage from Descartes's *Dioptrics*[44]

If the two small plates, AE and LF are perpendicular on the plank, the ray AL, passing through the two pinholes must be parallel to the plank EF. The angle RQP is a right angle, while the angle QRP is more acute than RPQ. The three sides of the prism are completely flat and polished, RQ is placed in contact with LF, and QP in contact with FP. The ray of light AL passes through B without undergoing any refraction. When it comes in contact with the oblique surface RP, it is refracted to some point on the plank (I).

In Descartes later words, "[…] the entire use of this instrument consists only in thus making the ray of sunlight pass through these holes A and L in order to know, by this means, the relation which the point I (that is, the center of the small oval of light that this ray describes on the on the plank EFI) has with the other two points B and P".[45] For this step, only material precision is important. Provided one follows the instructions, the plane LBPI would be perpendicular on the plane described by the plank, and any right angle glass prism would generate the same proportion. Points I, B, and P are important for the next stage, in which the precise anaclastic line is drawn.

2. The geometric stage

This stage is crucial for tracing the needed hyperbola. Initially, Descartes did not include this into the description of the machine sent to Ferrier. Not being a skillful mathematician, Ferrier writes to Descartes that he needs to know the secret by which one can trace the anaclastic line (the hyperbola, in this case) by means of ruler and compass. He also writes that Mydorge takes credit for being the only one to know how to trace this curve in such a way that the glass will not lose any thickness and its diameter will remain the same.[46] In his reply, Descartes includes detailed instructions on how to draw such a curve.

It is important to note that this procedure includes an application of the sine law of refraction.[47] Descartes probably discovered the law in 1626, but it was only fully described in the *Dioptrics* of 1637.[48] As he revealed to Mersenne two years later, he considered that Ferrier could take credit for the sine law of refraction.[49] Gaukroger thinks that this is the reason why Descartes did not present the detailed law, but only a "rule of thumb" that Ferrier could use. Still, the law could have been easily deduced from the "rule of thumb", and in his response from 1629, Descartes writes that he needs to think of a way in which this procedure will be clear once and for all.[50] So, it might not be fear that drove Descartes to present the procedure the way he did, but rather an attempt to convey this knowledge in a way in which an artisan would be able to comprehend it without a deep knowledge of geometry. Going back to this stage, the first thing that needs to be done is transfer the setup of stage one onto a paper:

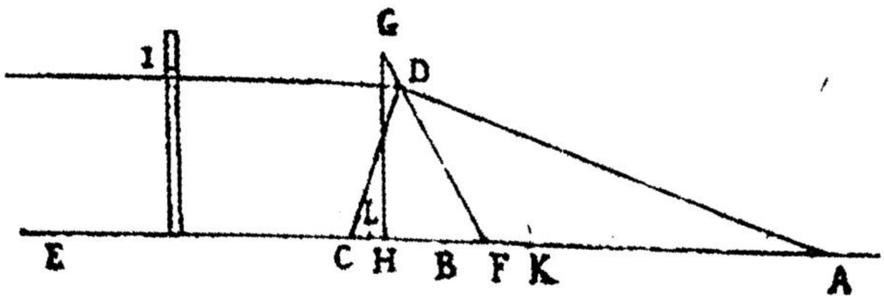


Illustration of the second stage[51]

The plank is defined by the points A and E and the glass prism is represented by FGH. As we saw, GH has to be perpendicular to AE. D is the point where the ray is refracted until A. From D, a line DC is drawn such that the angle FDC is equal to the angle ADF. The point in which DC intersects EA is C. From C, a line CK is drawn which is equal to CD. On AE, point L is marked such that AL=AD. The middle between K and L is marked B. Having the three points A, B and C, one can obtain the proportion between AB and BC, which is always constant, given that any considered is of the same transparent material.

This is important for my argument. It seems that the material object contains the thing that the mixed-mathematician ought to study: the proportion between the three points. This issue can be elusive. Descartes is able to apply the geometry onto the glass prisms. This would be the interpretation given within the Aristotelian subalternation framework. Let us stick to it for a while. On the one hand, if one changes the angle of the prism – makes angle FGH bigger – the points displace but the proportion between A, B and C is the same. The geometrical way of finding the proportion is, after all, the principle borrowed from mathematics. The procedure should be applied by the process of abstraction. Thus, regardless of the material *substratum* of the prism, the proportion between A, B and C should be the same, and should stay the same with each change of the angle FGH. On the other hand, if one changes the glass prism to a crystal one and does not alter the angle FGH, the former proportion would not be there at all, given that the angle ADF would be different. Still, the geometrical way in which one establishes the proportion in the case of the crystal prism is the same. This implies another process of abstraction, in which only the initial geometrical setup is different. This second process of abstraction also ignores the material *substratum*. However, Descartes wanted to establish precisely this: a property of the material *substratum*. He writes that: "[…] this proportion will be similar every time, any glass triangle you would choose, *provided that they are all made of the same glass*"[52]. It means that his *modus considerandi* need not account for the fact that the matter may have accidental properties which are not dealt with by the geometrical construal of the problem. Of course, the piece of glass may not be uniform. But, as the lens itself would be made of it, any accidental imperfection would be exhausted by the transfer of the setup of stage one onto paper. A Proclean interpretation fits this geometric stage better. The sought proportion between A, B and C is of a higher generality than the geometrical procedure for finding it. If this proportion is embedded into the material *substratum*, then it would project the property of the index of refraction, and would do so each time the transparent medium is different.

This proportion allows the pointwise construction of a hyperbola in the given frame of reference. If the prism is of the same glass as the future lens, the generated hyperbola will be proportional to the size of the prism. Thus, the lens will not lose any thickness and its diameter will be the largest one possible. Let us see how this construction is made:

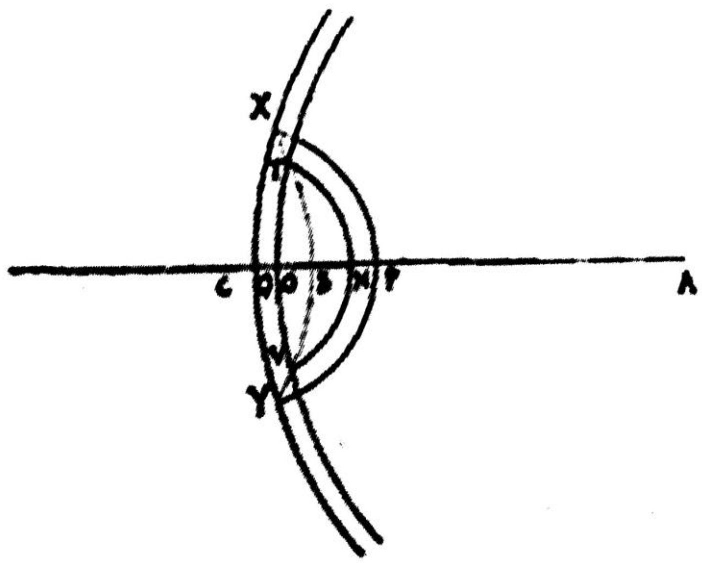


Illustration of the procedure for drawing the anaclastic hyperbola[53]

Probably because the drawing of the hyperbola requires some auxiliary constructions, Descartes pictured them in separate illustrations. Still, the frame of reference is the same. Having A, B and C, one can draw the anaclastic line in the following way: The tip of the compass is placed on point B. The compass may be opened as much as desired, and points N and O are traced on AC. Then, the two tips of the compass need to be placed on A and O. An arc TOV is traced. Moving the compass on C and N respectively, another arc is drawn, which needs to cut the first in T and V. The anaclastic line is now defined by the three points V, B and T. This procedure can be repeated *ad infinitum*. For instance, two other points equally distant from B can be drawn with the compass: Q and P. The intersections between the arcs constructed from A and C to P and Q respectively will determine two other points on the curve XTBVY. Thus, the hyperbola can be constructed point by point, the only limitation is the intelligibility of the drawing and the patience of the practitioner. After explaining this procedure, Descartes wrote to Ferrier "You made me laugh by calling this a secret; you would have found it yourself, if you really understood the things I presented before"[54], and then explains that this rule is nothing else than a complex rule of three.

We should acknowledge, however, that this type of construction of a complex curve was not regarded as acceptable in geometry. Even if the curve is traced using only the compass, it was not by a single continuous motion, but by independent and discontinuous motions of the compass. The procedure was not exact. Still, pointwise constructions had their place in practical geometry textbooks, and even Clavius used such constructions in his attempts to trisect an angle by a special curve.[55] Perhaps this is the reason why Descartes was explicit about not providing the full geometrical Demonstration in his Dioptrics.[56]

The next step in defining the lens-cutting machine was determined by the geometric stage. If one was to shape the lenses by the anaclastic line, the machine had to mechanically replicate the exact hyperbola which was established geometrically.

3. The mechanical stage

The mechanical stage needs to apply the result of the geometrical stage such that it makes the actual construction of the machine viable. In the same letter to Ferrier, Descartes considers this to be more difficult:

> But an even bigger secret is, having the three points A, B, C, […] or similar ones, to use them for finding the best angle on inclination which your machine should have; and I do not know of anyone else who would be able to determine it, although its employment is not difficult"[57].

It is more difficult because Descartes's lens-cutting machine does not actually grind lenses directly. It uses a file to shape and sharpen some blades which then do the job. At the end of this stage, the blades need to have the converse anaclastic shape. The remarkable thing is that Descartes also describes this stage in the same frame of reference, even though it is not geometrically connected to the previous one:

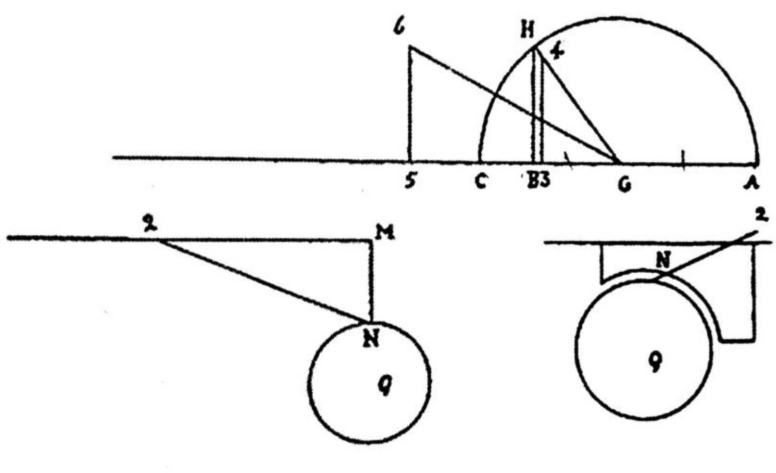
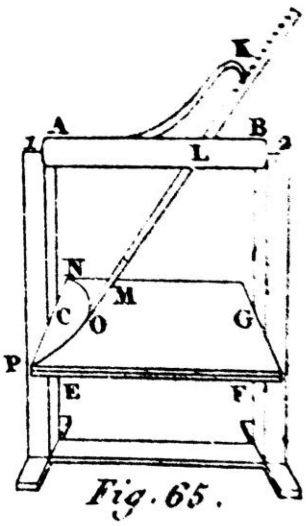


Illustration of the blades' angle of inclination[58]

To find the proper angle of the blade, the middle point between A and C must be traced. This is G. From G, a circle is drawn to A and C. From B, a perpendicular is raised that intersects the circle in H. HG is traced, and the angle HGB, along with its complement, HGA, is the needed angle. This should allow one to trace the Hyperbola "at a stroke, as, with a compass, we describe circles".[59] How does this solve the problem? Well, if we imagine the angle HGB to revolve around GB and if, from H, we imagine a plane parallel with the plank AC, the line HG will describe a section of a cone, which is exactly the same – in size and curvature – to the one previously defined by pointwise construction. Having all these steps precisely described, the design of the entire machine can be described:



The sketch of the lens-cutting machine[60]

The arm MK rotates around the axis AB. In its motion, MK intersects with the plank PNG by the line NOP. The blades are inserted horizontally within the plank PNG, and the movement of MK imposes upon them the shape of the anaclastic hyperbola. This is the setup described in the mechanical stage, only it is upside-down. Having several such blades defined in this fashion, the artisan can use a complex lathe to grind the lenses in the anaclastic shape of the hyperbola.

The purpose of the mechanical stage was to mechanically replicate the exact hyperbola traced in the second stage. On the Aristotelian subalternation model, this procedure does not belong to the optics. It is no longer about the behavior of the rays of light. Instead, this mechanical mixed mathematics should also work by borrowing geometrical principles. Transferring knowledge from the optics can only be done – due to the genera-crossing constraint – via the geometrical principles. But this cannot be the case, as mechanical tracing of cone sections was not acceptable in pure geometry. The motion that produces the curve mechanically was not a single motion. However, Descartes defined the three stages in the same frame of reference. He considered all of them as a single problem, not as three distinct problems solved by distinct mathematical apparatus. This is where a Proclean account would fill the gap. If the lens-grinding machine is seen as a projection of the law of refraction, the two unsound geometrical constructions – the pointwise hyperbola of the geometric stage and the mechanical cone-section of the mechanical stage – would be connected by the material proportion established by the law of refraction. This would enable the final construction of the machine to be of the same degree of certainty as the law of refraction itself.

On a theoretical level, the machine is, as Burnett argues, not meant to be a physical one, but a heuristic ideal device: "Conflating the physical machine and the heuristic <<ideal-machine,>> Descartes' device became an ideal system of linked components powered to execute his task".[61] Nevertheless, Descartes believed in the practicality of the endeavor. In the *Dioptrics*, after presenting all the mathematical and mechanical features of this machine, Descartes ends his description with a long discussion about the material and artisanal issues of its construction.[62] After all, the intended audience of the *Dioptrics* was not composed of philosophers, but rather artisans and practical mathematicians. Descartes believed that constructing this machine is an achievable task, and even thought that the manufacturer would obtain a long-term profit.[63]

III. Conclusion

In this paper, I argued that a Proclean interpretation of Descartes's attempt to build a lens-cutting machine raises fewer problems than an interpretation based on Aristotle's division of sciences.

First, on a Proclean reading, the design of the lens-cutting machine would not inherit the problem raised by genera-crossing. Proclus does not regard mathematical objects as abstractions which are fundamentally different than natural objects, but as gradual projections imposed by the higher forms of being. As such, Descartes's *modus considerandi* in the material stage need not account for the accidental character of the matter which is considered. Actually, Descartes treats this stage as accounting for a real property of the transparent medium. Aside from this property, no other accident in matter would affect the results. Also, Descartes is able to transfer the knowledge obtained in the material stage of the design to the geometrical and the mechanical stages. If pure geometry would be the only science which would provide principles to mixed-mathematics, the inferences between the stages would be grounded solely on pure geometry. But as we have seen in the geometric and mechanical stages, neither the pointwise construction nor the mechanical conic section were exact means of geometrical construction. In the design of the lens-cutting machine, the optical part was not considered as (i) only bound to the principles of geometry.

Second, mixed-mathematical sciences are, according to Proclus, projections ultimately given by the principles of the general mathematics. The final step in the construction of the machine is attainable with the same level of certainty as the previous stages. In this ideal way, the machine itself may be seen as a projection of the "rule of thumb" law of refraction of the geometric stage, combined with the established proportion for each transparent medium. If this is so, then there is a kind of heuristic common to both dioptrics and mechanics, and this heuristic can (ii) allow for conjoining different types of mixed-mathematics in a way in which certainty is preserved. If we suppose that the same higher general mathematics projects this heuristic, then, on a Proclean model, Descartes's ideal machine could be, in itself, a distinct branch of mixed-mathematics.

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[1] See Gaukroger, S., *Descartes – An Intellectual Biography*, (Oxford: Clarendon Press, 1995), 55-59.

[2] In a letter to Mersenne, in 1638, AT II 268.

[3] In a letter from 1639, AT II 542.

[4] References to Descartes's texts are given from the standard edition: Descartes, R., Œuvres de Descartes (hereafter AT) eds. Ch. Adam and P. Tannery, 2nd ed. (Paris: Vrin, 1964– 1974), 11 vols. The first mention of *Le Monde* is found in AT I 70. See also Gaukroger, (1995): 226-290.

[5] See Cantù, P., "Aristotle's prohibition rule on kind-crossing and the definition of mathematics as a science of quantities", Synthese 174 (2010): 225-235; Rabouin, D., "Mathematics and Imagination in Early Modern Times: Descartes and Leibniz' mathesis universalis in the light of Proclus' Commentary of Euclid's Elements" in *Knowledge and the Power of Imagination, 17th-18th Centuries*, ed. K. Vermeir (preprint, forthcoming)

[6] Sasaki, C., *Descartes' Mathematical Thought*, (Dordrecht: Springer Science+Business Media, 2003), especially chapters 1, 2, 7, 8 (pp. 13-95; 333-394).

[7] See Schuster, J., *Descartes-Agonistes: Physico-mathematics, Method & Corpuscular-Mechanism, 1618-33*, (Dordrecht: Springer Science+Business Media, 2013), chapter 5 (225-262).

[8] Hattab, H., "Descartes on the Eternal Truths and Essences of Mathematics: An Alternative Reading", *Vivarium* 54 (2016): 204-249.

[9] Schuster, J., "Consuming and Appropriating Practical Mathematics and the Mixed Mathematical Fields, or Being "Influenced" by Them: The Case of the Young Descartes", in *Mathematical Practitioners and the Transformation of Natural Knowledge in Early Modern Europe*, eds. L. Cormack, S. Walton, J. Schuster (Dordrecht: Springer, 2017), 37-65.
[10] Schuster, J., (2017): 63. Schuster mentions that the latter point about the lens-grinding machine was informally suggested to him by Michael S. Mahoney.
[11] Biener, Z., *The Unity of Science in Early-Modern Philosophy: Subalternation, Metaphysics and the Geometrical Manner in Scholasticism, Galileo and Descartes*, (Doctoral Dissertation, University of Pittsburg, 2008).
[12] Aristotle, *Posterior Analytics*, I.7 75a38-b10.
[13] Aristotle, *Posterior Analytics*, I.7.75b.9.
[14] Ian Mueller has proposed the historiographical label of "abstractionism" to account for Aristotle's view on mathematical objects and their ontological and epistemological status. The view has been nuanced and refined by later ancient commentators, most notably by Alexander of Aphrodisias. See Mueller, I., "Foreword to the 1992 Edition", in *Proclus, A Commentary on the First Book of Euclid's Elements*, trans. G. R. Morrow (Princeton: Princeton University Press, 1992), IX-LXVII; Mueller, I. "Aristotle's doctrine of abstraction in the commentators", in *Aristotle Transformed – The Ancient Commentators and their Influence*, ed. R. Sorabji (New York: Cornell University Press, 1990), 463-481; Mueller, I., "Aristotle on geometric objects", eds. J. Barnes, M. Schofield, R. Sorabji, *Articles on Aristotle*, vol. 3, (London: Duckworth 1979), 96-107; Mueller, I., "Iamblichus and Proclus' Euclid commentary", Hermes 115 (1987): 334-348.
[15] Aristotle, *Posterior Analytics*, 85a31-85b3.
[16] Jardine, N., "Demonstration, Dialectic and Rhetoric in Galileo's Dialogue", in *The Shapes of Knowledge from the Renaissance to the Enlightenment*, eds. D. Kelley, R. Popkin (Dordrecht: Kluwer Academic Publishers, 1991), 102.
[17] McKirahan, R. D., "Aristotle's Subordinate Sciences", *British Journal for the History of Science* 11 (1978): 197–220.
[18] Kelley, D. R., (ed.) *History and the Disciplines: The Reclassification of Knowledge in Early Modern Europe* (Rochester: University of Rochester Press, 1997).
[19] Biener, Z., (2008) This is the argument that Biener advances. He treats the subalternated character of mixed-mathematics via the concept of tempered-hylomorphism and argues that subalternation extended beyond the original Aristotelian sense.
[20] Mueller, I., (1992).
[21] See parts one and two of the Prologue, in *Proclus, A Commentary on the First Book of Euclid's Elements*, trans. G. R. Morrow (Princeton: Princeton University Press, 1992), 3-58. Hereafter I will reference Proclus's text section, followed by the page number in Morrow's translation.
[22] Proclus, (1992): 15, pp. 12-13.
[23] Proclus, (1992): 4, p. 3.
[24] Proclus, (1992): 3-7, pp. 3-5.
[25] See Mueller, I., (1992): lxi-lxiv; See also Claessens, G. "Imagination as Self-knowledge: Kepler on Proclus' *Commentary on the First Book of Euclid's Elements*", *Early Science and Medicine* 16 (2011): 179-199;
[26] Mueller, I., (1992): xvii
[27] Proclus, (1992): 19-20, p. 17.
[28] Proclus, (1992): 22, p. 19.
[29] Clavius, Opera mathematica 1:3, in Lattis, J., *Between Copernicus and Galileo: Cristoph Clavius and the Collapse of Ptolemaic Cosmology*, (Chicago: University of Chicago Press, 1994), 36.
[30] Burnett, D. G., "Descartes and the Hyperbolic Quest: Lens Making Machines and Their Significance in the Seventeenth Century", *Transactions of the American Philosophical Society*, 95/3

(2005): 1-152. Here, Burnett undermines the distinction between mathematicians, mathematical practitioners and artisans. Still, for the purposes of this paper, the difference between Descartes and Ferrier is best explained in terms of the type of knowledge they employed in the optics.

[31] Mersenne, M., *La verité des Science Contra les Septiques ou Pyrroniens*, (1625): 229-230.

[32] See Dijksterhuis, F., *Lenses and Waves: Christiaan Huygens and the Mathematical Science of Optics in the Seventeenth Century*, (New York: Kluwer Academic Publishers, 2004): 1-35. Also, see Burnett, D. G., (2005): 1-19.

[33] See Willach, R., "The long road to the invention of the telescope" in *The origins of the telescope*, eds. A. Van Helden, S. Dupré, R. van Gent, H. Zuidervaart (Amsterdam: KNAW Press, 2010): 93-114; also Burnett D. G., (2005): 5-6.

[34] Burnett, D. G., (2005): 7.

[35] For an overview of the improvements in telescope production techniques see Van Helden, "Invention of the Telescope." in *Transactions of the American Philosophical Society* 67/4 (1977), pp. 1-67

[36] It was not until Newton that this type of aberration was clearly defined and approached as a clear-cut limitation. See Burnett D. G., (2005): 17.

[37] Several of Descartes's correspondents, such as Marin Mersenne, Claude Mydorge or Isaac Beeckman were interested in dioptrics and the aplanatic properties of lenses.

[38] The letter exchanges with Ferrier are found in: AT I 13-16; AT I 32-37; AT I 38-52; AT I 53-69; AT I 183-187. For the *Dioptrics*, I will use the translation by Olscamp, P. J., *Discourse on Method, Optics, Geometry, and Meteorology*, (Indianapolis: Hackett Publishing Company, 2001).

[39] The habit was that the artisans had to be constantly directed by the employers. A good example here is Galileo. See Burnett, D. G., (2005): 20-71.

[40] A detailed account of their interaction is provided by William Shea. See Shea, W., "Descartes and the French Artisan Ferrier", *Annali dell'Istituto e Museo di storia della scienza di Firenze*, 7/2 (1982): 145-160.

[41] AT I 13, my translation.

[42] AT I 50.

[43] AT I 62-64.

[44] AT VI 216. This illustration was given in the 1637 *Dioptrics*. Descartes did not include one in his reply to Ferrier. However, the anachronistic use of the illustration here does not affect my argument, as, in his reply, Descartes describes in detail the setup that needs to be put into place for establishing the index of refraction.

[45] AT VI 217-218; Olscamp, P. J. (2001): 163.

[46] AT I 51.

[47] AT I 63-66.

[48] Gaukroger and Burnett think the law was discovered in 1626. See Gaukroger, S., (1995): 139-146; Burnett, D. G., (2005): 16.

[49] AT I 228.

[50] AT I 63.

[51] AT I 63.

[52] AT I 63; my translation, my emphasis.

[53] AT I 63.

[54] AT I 63; my translation.

[55] Sasaki, C., (2003): 70.

[56] AT VI 221-222; Olscamp, P. J. (2001): 171.

[57] AT I 67; my translation.

[58] AT I 67.

[59] AT VI 219; Olscamp, P. J. (2001): 164-165.
[60] AT VI 219; Olscamp, P. J. (2001): 165.
[61] Burnett, D. G., (2005): 124.
[62] See AT VI 219-223; Olscamp, P. J. (2001): 167-173.
[63] See AT I 33.